Power Waves and the Scattering Matrix

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Abstract—This paper discusses the physical meaning and properties of the waves defined by

$$a_i = \frac{V_i + Z_i I_i}{2\sqrt{|\text{Re } Z_i|}}, \qquad b_i = \frac{V_i - Z_i^* I_i}{2\sqrt{|\text{Re } Z_i|}}$$

where V_i and I_i are the voltage at and the current flowing into the ith port of a junction and Z_i is the impedance of the circuit connected to the ith port. The square of the magnitude of these waves is directly related to the exchangeable power of a source and the reflected power. For this reason, in this paper, they are called the power waves. For certain applications where the power relations are of main concern, the power waves are more suitable quantities than the conventional traveling waves. The lossless and reciprocal conditions as well as the frequency characteristics of the scattering matrix are presented.

Then, the formula is given for a new scattering matrix when the Z_i's are changed. As an application, the condition under which an amplifier can be matched simultaneously at both input and output ports as well as the condition for the network to be unconditionally stable are given in terms of the scattering matrix components. Also a brief comparison is made between the traveling waves and the power waves.

I. Introduction

THE CONCEPT of traveling waves along a transmission line and the scattering matrix of a junction of transmission lines are well known and they play important roles in the theory of microwave circuits. However, the traveling wave concept is more closely related to the voltage or current along the line than to the power in a stationary state. If a circuit which terminates a line at the far end is not matched to the characteristic impedance of the line, even if the circuit has no source at all, we have to consider two waves traveling in opposite directions along the line. This makes the calculation of power twice as complicated. For this reason, when the main interest is in the power relation between various circuits in which the sources are uncorrelated, the traveling waves are not considered as the best independent variables to use for the analysis. A different concept of waves is introduced. The incident and reflected power waves a_i and b_i are defined by

$$a_i = \frac{V_i + Z_i I_i}{2\sqrt{|\text{Re } Z_i|}}, \qquad b_i = \frac{V_i - Z_i^* I_i}{2\sqrt{|\text{Re } Z_i|}} \tag{1}$$

where V_i and I_i are the voltage and the current flowing into the ith port of a junction and Z_i is the impedance looking out from the ith port. The positive real value is chosen for the square root in the denominators. These power waves were first introduced by Penfield [1][1] for the discussion of noise performance of negative resistance amplifiers and later they were used for the discussion of actual noise measure of linear amplifiers by Kurokawa [2]. However, since it was not their main objective, the meaning of these waves and the properties of the corresponding scattering matrix were only briefly discussed. At about the same time, Youla [3] studied the same waves; however, his Z_i's were limited to have positive real part only. More recently, Youla and Paterno used these waves to study the attenuation error in mismatched systems [4].

The purpose of this paper is to present the physical meaning of the waves defined by (1) as well as the properties of the scattering matrix based on this new wave concept. Some of the properties such as the lossless condition for the matrix have been discussed in the previous papers. However, for the sake of completeness, they are included in this paper also.

II. Physical Meaning

Since the waves defined by (1) are closely related with the exchangeable power [5] of a generator, we have to discuss briefly what it is. For this purpose, let us consider the equivalent circuit of a linear generator, as shown in Fig. 1, in which Z_i is the internal impedance and E_o is the open circuit voltage of the generator. The power P_L into a load Z_L is given by Re $Z_L|I_i|^2$, where I_i is the current into the load. Since the magnitude of the current is equal to $|E_o/(Z_L+Z_i)|$, P_L is given by

$$P_L = \text{Re } Z_L \left|\frac{E_o}{Z_L + Z_i}\right|^2 = \frac{R_L |E_o|^2}{(R_L + R_i)^2 + (X_L + X_i)^2} \tag{2}$$

$$= \frac{|E_o|^2}{4R_i + \dfrac{(R_L - R_i)^2}{R_L} + \dfrac{(X_L + X_i)^2}{R_L}} \tag{3}$$

where R_L and R_i are the real parts of Z_L and Z_i, respectively, and X_L and X_i are the imaginary parts. With $R_i > 0$, we can easily see from (3), that the denominator becomes minimum when

$$R_L = R_i, \qquad X_L = -X_i \tag{4}$$

The corresponding maximum power P_L is

$$P_a = \frac{|E_o|^2}{4R_i}, \qquad (R_i > 0) \tag{5}$$

Manuscript received September 8, 1964.
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[1] In the original definition, Re Z_i is taken instead of $|\text{Re } Z_i|$ in the square root of the denominator of (1) (cf Section VIII).

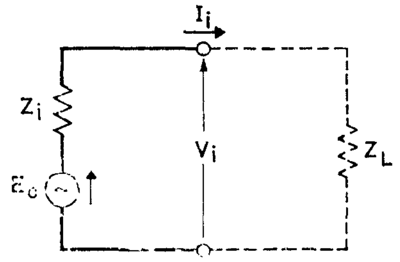

Fig. 1. Equivalent circuit of a linear generator.

This maximum power is called the available power of the generator. When the real part of Z_i is negative, P_L becomes infinite as R_L and X_L approach $-R_i$ and $-X_i$, respectively as we can see from (2). In this case, (5) no longer represents the maximum power that can be drawn from the generator. However, the expression given in (5) remains finite and the power represented by it is called the exchangeable power P_e of the generator, for any nonzero R_i. That is,

$$P_e = \frac{|E_o|^2}{4R_i}, \qquad (R_i \lessgtr 0) \tag{6}$$

Thus, for $R_i > 0$, the exchangeable power is the maximum power that the generator can supply. With $R_i < 0$, the exchangeable power is no longer equal to the maximum possible power flow into the load, which is infinite. However, regardless of the sign of R_i it can be considered as the stationary value of the expression P_L with respect to a small change of the load impedance Z_L. This can be easily seen from (3), in which R_L and X_L appear only in the second-order terms of the difference between R_L and R_i and of the difference between X_L and $-X_i$.

Now, we are in a position to discuss the waves defined by (1). In the discussion of electric circuits, the voltage and current at the terminals are generally chosen as the independent variables. However, one may equally well choose any linear transformation of them as long as the transformation is not singular, i.e., as long as the inverse transformation exists. The waves defined by (1) are the result of just one of an infinite number of such linear transformations.

With a fixed Z_i, if V_i and I_i are given, a_i and b_i are readily calculated from (1). On the other hand, if a_i and b_i are given, V_i and I_i are obtained from the inverse transformation

$$V_i = \frac{p_i}{\sqrt{|\operatorname{Re} Z_i|}} (Z_i^* a_i + Z_i b_i),$$

$$I_i = \frac{p_i}{\sqrt{|\operatorname{Re} Z_i|}} (a_i - b_i) \tag{7}$$

where p_i is defined by

$$p_i = \begin{cases} 1 & \text{when } \operatorname{Re} Z_i > 0 \\ -1 & \text{when } \operatorname{Re} Z_i < 0 \end{cases} \tag{8}$$

Thus, any result in terms of one set of variables can easily be converted to that in terms of the other set of variables. This justifies the use of the waves a_i and b_i defined by (1) in place of the terminal voltage and current for any analysis. Referring to Fig. 1, the voltage at the generator terminal is given by

$$V_i = E_o - Z_i I_i$$

Inserting this into the first expression in (1), and taking the square of the magnitude, we have

$$|a_i|^2 = \frac{|E_o|^2}{4|R_i|}$$

which is equivalent to

$$P_e = p_i |a_i|^2 \tag{9}$$

It is worth noting that, when E_0 is equal to zero, a_i becomes zero also.

Next, let us consider $|a_i|^2 - |b_i|^2$. Direct substitution of (1) into this expression gives

$$|a_i|^2 - |b_i|^2$$
$$= \frac{(V_i + Z_i I_i)(V_i^* + Z_i^* I_i^*) - (V_i - Z_i^* I_i)(V_i^* - Z_i I_i^*)}{4|R_i|}$$
$$= \frac{(Z_i + Z_i^*)(V_i I_i^* + V_i^* I_i)}{4|R_i|} = p_i \operatorname{Re}\{V_i I_i^*\}$$

from which we have

$$\operatorname{Re}\{V_i I_i^*\} = p_i(|a_i|^2 - |b_i|^2) \tag{10}$$

The left-hand side of (10) expresses the power which is actually transferred from the generator to the load. Therefore, this is called the actual power from the generator (or to the load). Equation (10) shows that the actual power is equal to $p_i(|a_i|^2 - |b_i|^2)$. Since $-|b_i|^2$ is always negative whether the load contains some source or not, the magnitude of the exchangeable power of a generator $|a_i|^2$ can be identified as the maximum power that the generator can supply when $R_i > 0$, and as the maximum power that the generator can absorb when $R_i < 0$.

For a moment, let us confine ourselves to the case where the real part of the internal impedance of the generator is positive, i.e., p_i is equal to 1. Then, (9) and (10) can be interpreted as follows. The generator is sending the power $|a_i|^2$ toward a load, regardless of the load impedance. However, when the load is not matched, i.e., if (4) is not satisfied, a part of the incident power is reflected back to the generator. This reflected power is given by $|b_i|^2$ so that the net power absorbed in the load is equal to $|a_i|^2 - |b_i|^2$. Associated with these incident and reflected powers, there are waves a_i and b_i, respectively.

To help understand the meaning of the incident and reflected powers, let us consider a new equivalent circuit of the generator in which we see these powers separately. Suppose that a new generator and load are connected to two arms of a three-port circulator and they are matched to the circulator impedance and that a lossless circuit which transforms the circulator impedance into Z_i is connected, as shown in Fig. 2. The maximum power we can obtain from the third arm of the circulator is equal to the power the new generator supplies toward the circulator. Because the lossless circuit does not consume any power, this maximum power must be equal to the maximum power which the outside load Z_L can absorb. Since the change of the load impedance Z_L does not affect the load condition of the generator at arm 1, the available power $|a_i|^2$ must be equal to the power which the generator is sending to the circulator. Further, since the net power to the load Z_L is equal to $|a_i|^2-|b_i|^2$ and as no power comes back to the generator at arm 1, the balance $|b_i|^2$ must be absorbed in the load connected to arm 2 of the circulator. Thus we see that the incident power from the original generator is the power which the internal generator in this equivalent circuit is producing and the reflected power is the power which the internal load is absorbing.

Since one may well argue that, using an arbitrary constant C, $|a_i|^2+C$ is the incident power from a generator while $|b_i|^2+C$ is the reflected power, the above interpretation of incident and reflected powers is somewhat arbitrary. However, we set C equal to zero so that the maximum power a load can absorb is equal to the incident power which the generator sends to the load. This situation is very similar to that of the Poynting vector $E\times H$. Using an arbitrary vector function X, $E\times H+\nabla\times X$ can be considered as the transmission power density; whenever it is integrated over a closed surface the contribution from the last term $\nabla\times X$ disappears. Nevertheless, we generally consider that the power density is expressed by $E\times H$, so that there is no energy flow where there is no electric or magnetic field.

Extending our discussion to the case where the real part of the internal impedance of the generator may be negative, we say that the generator is sending the power $p_i|a_i|^2$ toward the load regardless of the load impedance and, when the load is not matched, $p_i|b_i|^2$ is reflected back so that the net power absorbed in the load is given by $p_i(|a_i|^2-|b_i|^2)$. Associated with these incident and reflected powers, there are the incident and reflected waves a_i and b_i. Since the incident power to a load is equal to the exchangeable power of the generator connected to the load, $p_i|a_i|^2$ may also be called the exchangeable power to the load. The reason why, for the discussion of powers, we do not consider the incident and reflected powers directly but through the waves a_i and b_i lies in the fact that there is a linear relation between a_i's and b_i's and this can be used advantageously as we shall see in the following sections. There is no such relation between powers.

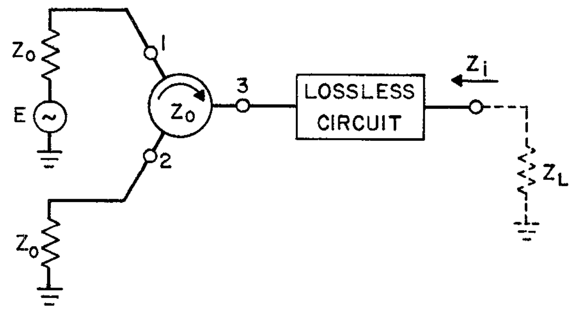

Fig. 2. New equivalent circuit of a generator.

III. Reflection Coefficients and Scattering Matrix

When we consider two quantities such as voltage and current, we take the ratio, an impedance. Similarly, since we have two quantities a_i and b_i, let us define the ratio s

$$s=\frac{b_i}{a_i} \tag{11}$$

and call it the power wave reflection coefficient.[2] Further, let us call the square of its magnitude, i.e., $|s|^2$, the power reflection coefficient. Using (1) and the relation $V_i=Z_LI_i$, s can be expressed in terms of impedances.

$$s=\frac{Z_L-Z_i^*}{Z_L+Z_i} \tag{12}$$

Substituting $Z_i=R_i+jX_i$, $Z_L=R_L+jX_L$ into (12), s can be rewritten in the form

$$s=\frac{R_L+j(X_L+X_i)-R_i}{R_L+j(X_L+X_i)+R_i} \tag{13}$$

Comparing this expression with that of the conventional voltage reflection coefficient, we see that s corresponds to the vector drawn from the center of the Smith chart to the point where the normalized impedance is given by $[R_L+j(X_L+X_i)]/R_i$. In other words, if the reactance part of Z_i is added to Z_L and normalized with respect to the real part of Z_i, then the corresponding point on the Smith chart gives the magnitude and the phase of the power wave reflection coefficient. From this, the following important property of s is derived: When R_i and R_L have the same sign, $|s|<1$ and when they have opposite signs, $|s|>1$.

The power reflection coefficient is given by

$$|s|^2=\left|\frac{Z_L-Z_i^*}{Z_L+Z_i}\right|^2 \tag{14}$$

When the matching condition (4) is satisfied, the power reflection coefficient becomes zero, as is expected.

[2] When Z_i is real and positive, this is a voltage reflection coefficient.

s and $|s|^2$ are the reflection coefficients looking into the load from the generator side. The corresponding reflection coefficients s' and $|s'|^2$ looking into the generator from the load must be given by

$$s' = \frac{Z_i - Z_L^*}{Z_i + Z_L}, \qquad |s'|^2 = \left|\frac{Z_i - Z_L^*}{Z_i + Z_L}\right|^2$$

where the subscripts i and L are interchanged. s' is not necessarily equal to s. However, since $|Z_i - Z_L^*| = |Z_i^* - Z_L| = |Z_L - Z_i^*|$, $|s'|^2$ is always equal to $|s|^2$. Thus the power reflection coefficient remains the same when roles of generator and load are interchanged.[3] $1-|s|^2$ is called the power transmission coefficient and this also remains constant when we interchange the role of generator and load. It is worth noting that the power transmission coefficient times the exchangeable power is equal to the actual power, or that the actual power divided by the power transmission coefficient is the exchangeable power.

Next, to define the scattering matrix, let us consider a linear n-port network and let a, b, v and i be vectors whose ith components are a_i, b_i, V_i, and I_i at the ith port of the network respectively. Then, a and b can be written in terms of v and i as follows:

$$a = F(v + Gi), \qquad b = F(v - G^+ i) \tag{15}$$

where F and G are the diagonal matrices whose ith diagonal components are given by $1/2\sqrt{|\text{Re}Z_i|}$ and Z_i, respectively, and $^+$ indicates, in general, the complex conjugate transposed matrix. Since there is a linear relation between v and i given by

$$v = Zi \tag{16}$$

where Z is the impedance matrix, and since a and b are the result of a linear transformation of v and i, there must be a linear relation between a and b. Let us write it in the form

$$b = Sa \tag{17}$$

and call this S the power wave scattering matrix. Elimination of a, b, and v from (15), (16), and (17) gives

$$F(Z - G^+)i = SF(Z + G)i$$

from which the following expression of S can be obtained.

$$S = F(Z - G^+)(Z + G)^{-1}F^{-1} \tag{18}$$

Similarly, Z can be expressed in terms of S

$$Z = F^{-1}(I - S)^{-1}(SG + G^+)F \tag{19}$$

where I is a unit matrix. In Sections IV and V, we shall consider the conditions which S has to satisfy in order to represent a reciprocal network and a lossless network, respectively.

[3] In this interchange, only the place of the (zero-impedance) voltage source and the (zero-impedance) load current meter are reversed, leaving the generator and load impedance stationary.

IV. Reciprocal Condition

It is a well-known fact that the impedance matrix Z representing a reciprocal network has to satisfy the relation

$$Z = Z_t \tag{20}$$

where the subscript t indicates the transposed matrix. The corresponding relation for S is given by

$$S_t = PSP \tag{21}$$

where P is a diagonal matrix with its ith diagonal component being p_i. The proof of (21) will be given in Appendix I. Equation (21) is equivalent to

$$S_{ji} = p_i p_j S_{ij} \tag{22}$$

which implies that, if the signs of Re Z_i and Re Z_j are the same, S_{ij} is equal to S_{ji} and, if they are opposite, S_{ij} is equal to $-S_{ji}$. For either case

$$|S_{ij}|^2 = |S_{ji}|^2 \tag{23}$$

Now, suppose that all the circuits except the one connected to the ith port of the network have no source. Since the power from the jth circuit to the network is generally given by $p_j(|a_j|^2 - |b_j|^2)$ and $a_j (j \neq i)$ is equal to zero, the power to the jth circuit from the network is given by $p_j|b_j|^2$. Further, in this case, b_j is equal to $S_{ji}a_i$ and hence, the ratio of the actual power $p_j|b_j|^2$ into the load j to the exchangeable power $p_i|a_i|^2$ from the source i is equal to $p_i p_j |S_{ji}|^2$. However, because of (23), the value of this ratio does not change when the subscripts i and j are interchanged. Thus, we conclude that the relation between the actual power into a load and the exchangeable power from the source stays constant when the roles of source and load are interchanged in a reciprocal network. This is a power reciprocal relation.

It is interesting to note that there is no such reciprocal theorem in general between the exchangeable power to a load and the exchangeable power from the source, nor between the actual power to a load and the actual power from the source. The actual power into the jth circuit is given by $p_j|b_j|^2$ and the power transmission coefficient at this point by $1-|S_{jj}|^2$. Therefore, the exchangeable power to the jth circuit is equal to $p_j|b_j|^2/(1-|S_{jj}|^2)$. The ratio of the exchangeable power into the jth circuit to that from the ith circuit is given by $p_j p_i |S_{ji}|^2/(1-|S_{jj}|^2)$. However, since $|S_{jj}|$ is not necessarily equal to $|S_{ii}|$, the value of this ratio does not necessarily stay constant when the roles of source and load are interchanged. Similarly, since the actual power from the ith circuit is given by $p_i|a_i|^2(1-|S_{ii}|^2)$, the ratio of the actual power into the jth circuit to that from the ith circuit is equal to $p_i p_j |S_{ji}|^2/(1-|S_{ii}|^2)$. This again does not remain constant when the subscripts i and j are interchanged. However, the ratio of the exchangeable power to the load j to the actual power from the source i is given by

$p_i p_j |S_{ji}|^2/(1-|S_{jj}|^2)(1-|S_{ii}|^2)$ and remains constant when the subscripts are interchanged.

The foregoing discussion is readily applicable to a pair of antennas. It is a well-known fact that there exists a power reciprocal relation between the transmitting and receiving antennas. However, it seems to be less well understood that it is between the exchangeable power from the source and the actual power to the load that the reciprocal theorem generally holds. Unless the matching conditions for both antennas are satisfied, the reciprocal theorem does not necessarily hold between the actual powers nor between the exchangeable powers.

V. Lossless Condition

In this section, let us consider the condition which a scattering matrix has to satisfy in order to represent a lossless network. The actual power into the network from the ith circuit is given by $p_i(|a_i|^2-|b_i|^2)$. Therefore, the total power into the network is

$$\sum_i p_i(|a_i|^2 - |b_i|^2).$$

When the network is lossless, this total power must be zero, hence we have

$$\sum_i p_i(|a_i|^2 - |b_i|^2) = 0$$

which we can rewrite in a matrix form as follows

$$a^+Pa - b^+Pb = 0$$

Substitution of (17) gives

$$a^+(P - S^+PS)a = 0$$

Since a is arbitrary, this means

$$S^+PS = P. \tag{24}$$

Equation (24) is the condition that the scattering matrix representing a lossless network has to satisfy.

For a simple example, let us consider a two-port junction. Equation (24) gives three independent conditions

$$\begin{aligned} p_1|S_{11}|^2 + p_2|S_{21}|^2 &= p_1 \\ p_1S_{11}S_{12}^* + p_2S_{21}S_{22}^* &= 0 \\ p_1|S_{12}|^2 + p_2|S_{22}|^2 &= p_2 \end{aligned} \tag{25}$$

From the second condition, we have

$$|S_{11}|^2|S_{12}|^2 = |S_{21}|^2|S_{22}|^2$$

Combining the first and last conditions in (25) with this equation, we obtain

$$\frac{p_2}{p_1}(1 - |S_{22}|^2)|S_{11}|^2 = \frac{p_1}{p_2}(1 - |S_{11}|^2)|S_{22}|^2$$

which is equivalent to

$$|S_{11}|^2 = |S_{22}|^2 \tag{26}$$

Equation (26) shows that, for a lossless two-port junction, the power reflection coefficient at one port is equal to that at the other port. From this conclusion, we see that the power reflection coefficient as well as the power transmission coefficient remain constant regardless of the position of the reference plane we take along a lossless transmission system. This means that we can choose any convenient plane as the reference plane for power discussion in a lossless transmission system.

The fact that the exchangeable power is preserved during a nonsingular lossless transformation can also be easily shown using the above result. Let a_2 be zero for a moment. The exchangeable power from the output port 2 is given by

$$\frac{p_2|b_2|^2}{1 - |S_{22}|^2} = \frac{p_2|S_{21}|^2|a_1|^2}{1 - |S_{11}|^2} = p_1|a_1|^2$$

where we have used the first condition in (25). The right-hand side of this equation is just the exchangeable power to the lossless junction. Thus, we have shown that the exchangeable powers are the same at the input and output of a lossless two-port junction provided that $|S_{11}|^2=|S_{22}|^2\neq 1$. If $|S_{11}|^2=|S_{22}|^2=1$, the input and output ports are effectively disconnected inside the junction, which is of no practical interest.

Inserting (26) back into the first and last expressions in (25), and comparing the result, we have

$$|S_{12}|^2 = |S_{21}|^2 \tag{27}$$

This is a kind of power reciprocal theorem. However, it is only the lossless condition that we have used for the derivation. Therefore, even a nonreciprocal twoport junction has to satisfy the power reciprocal theorem if it is lossless.

Coming back to (24), and multiplying both sides by $(PS)^{-1}=S^{-1}P^{-1}$ from the right and SP^{-1} from the left, we have

$$SPS^+ = P \tag{28}$$

Equations (24) and (28) are equivalent to each other. However, sometimes one may find (28) being more convenient than (24). The example is found in the discussion of the actual noise measure of linear amplifiers.

When the network is lossy, the total power into the network must be positive and hence

$$a^+(P - S^+PS)a \geq 0$$

Thus, for a passive network, $(P-S^+PS)$ must be positive definite or positive semidefinite.

VI. Frequency Characteristic of Lossless Reciprocal Network

When a junction under consideration is lossless as well as reciprocal, (21) and (24) must be satisfied simultaneously. Further in this case, corresponding to the well-known relation for $\partial Z/\partial\omega$,

$$i^+ \frac{\partial Z}{\partial \omega} i = j \int (\mu H^* \cdot H + \epsilon E^* \cdot E) dv \tag{29}$$

we can derive a relation for $\partial S/\partial \omega$ as we shall do in Appendix II. It is given by

$$a^+ j\left(K^+ + S^+K^+S + 2S^+P\frac{\partial S}{\partial \omega}\right)a = \int (\mu H^* \cdot H + \epsilon E^* \cdot E) dv \tag{30}$$

where K is a diagonal matrix with the ith diagonal component being

$$\left\{\frac{\partial}{\partial \omega}(Z_i^* - Z_i)\right\} \Big/ \left| Z_i^* + Z_i \right|,$$

and E and H are the electric and magnetic field respectively. The integral in the right-hand side of (30) extends all over the junction region and represents twice the stored energy in the network, hence it is positive. Thus, we see that

$$j\left(K^+ + S^+K^+S + 2S^+P\frac{\partial S}{\partial \omega}\right)$$

has to be positive definite. The first and second terms represent the effect of the possible change of the terminal impedance Z_i's. It is interesting to note that both terms disappear when all the imaginary parts of Z_i's remain constant. When this is the case, (30) reduces to

$$a^+\left(j2S^+P\frac{\partial S}{\partial \omega}\right)a = \int (\mu H^* \cdot H + \epsilon E^* \cdot E) dv \tag{31}$$

For a lossless oneport network, $|S_{11}| = 1$ and S_{11} can be written in the form $e^{-j\phi}$. Therefore, in this case the above relation reduces to

$$\frac{\partial \phi}{\partial \omega} = \frac{1}{2p_1 |a_1|^2} \int (\mu H^* \cdot H + \epsilon E^* \cdot E) dv$$

However, since $b_1 = e^{-j\phi}a_1$ and $\partial \phi/\partial \omega$ give the time delay between b_1 and a_1 of the wave envelope, which can be interpreted as the energy delay, the interpretation of (31), when applied to a one-port network, is as follows. The time required for an incident energy to enter the network and leave again is the total stored energy divided by the exchangeable power of the source. When the real part of the source impedance is negative, the required time becomes negative. This is what we expect if no oscillation takes place. In most cases where we connect a negative resistance to a oneport network of which the losses are negligible, oscillations occur and therefore it is impossible to observe the above phenomenon directly. For multiport networks, even if some of the impedances have negative real parts, a stable operation becomes possible and (30) with the corresponding p_i's being negative gives a more realistic condition for $\partial S/\partial \omega$.

One might think that the imaginary parts of the circuit impedances Z_i could be considered to be part of the junction and that K could therefore always be set equal to zero without loss of generality. This is not necessarily the case, for the imaginary parts of the circuit impedances Z_i may not have the frequency dependence of ordinary passive networks. Examples are $-L$ and $-C$.

VII. Change of Circuit Impedance

Suppose that the impedances of the circuits connected to the junction under consideration are changed from Z_i to $Z_i'(i = 1, 2, \cdots, n)$. Then the incident and reflected waves have to be redefined accordingly. The scattering matrix S' connecting these new power wave vectors is, of course, different from the original one. However, it is expressible in terms of the original S and the power wave reflection coefficient r_i of Z_i' with respect to Z_i^*, i.e.,

$$S' = A^{-1}(S - \Gamma^+)(I - \Gamma S)^{-1}A^+ \tag{32}$$

where Γ and A are the diagonal matrices with their ith diagonal components being r_i and $(1 - r_i^*)\sqrt{|1 - r_i r_i^*|}/|1 - r_i|$, respectively. An outline of the derivation is given in Appendix III. Essentially the same formula (for Re $Z_i > 0$) is also derived by Youla and Paterna [4] using a different approach.

There are a number of applications of this formula. Consider, for example, a twoport amplifier whose source and load impedances, Z_1 and Z_2, respectively, have positive real parts and let us obtain the condition under which both input and output ports can be matched simultaneously without changing the signs of the real parts of the source and/or load impedances. The matching conditions for input and output ports are given by $S_{11}' = 0$ and $S_{22}' = 0$, respectively. Using (32), the condition $S_{11}' = 0$ provides

$$r_1^* = \frac{S_{11} + r_2(S_{12}S_{21} - S_{11}S_{22})}{1 - r_2 S_{22}} \tag{33}$$

Similarly, $S_{22}' = 0$ provides

$$r_2^* = \frac{S_{22} + r_1(S_{12}S_{21} - S_{11}S_{22})}{1 - r_1 S_{11}} \tag{34}$$

For simultaneous matching, (33) and (34) have to be satisfied at the same time. Thus, the problem is reduced to that of finding the solutions of the simultaneous equations and checking whether or not they satisfy the appropriate conditions which ensure that the real parts of the source and load impedances remain positive. As explained in connection with (13), the latter conditions are given by $|r_1| < 1$ and $|r_2| < 1$, respectively. For the check of these conditions, a straightforward but lengthy calculation is necessary. From it, we see that when $|S_{12}S_{21}| \neq 0$ the necessary and sufficient condition for simultaneous matching to be possible is given by

$$2|S_{12}S_{21}| < 1 + |S_{12}S_{21} - S_{11}S_{22}|^2 - |S_{11}|^2 - |S_{22}|^2 \quad (35)$$

When $|S_{12}S_{21}| = 0$, the same condition is given by

$$|S_{11}| < 1 \quad \text{and} \quad |S_{22}| < 1 \quad (36)$$

The transducer gain under the simultaneously matched condition is

$$|S_{21}'|^2 = \frac{|S_{21}|}{|S_{12}|}(k \pm \sqrt{k^2 - 1}) \quad \text{for } |S_{12}S_{21}| \neq 0 \quad (37)$$

where

$$k = \frac{1 + |S_{12}S_{21} - S_{11}S_{22}|^2 - |S_{11}|^2 - |S_{22}|^2}{2|S_{12}S_{21}|} \quad (38)$$

The upper sign applies when

$$B = |S_{22}|^2 - |S_{11}|^2 - 1 + |S_{12}S_{21} - S_{11}S_{22}|^2 \quad (39)$$

is positive, and the lower sign when $B < 0$. When $|S_{12}S_{21}| = 0$, the same gain is

$$|S_{21}'|^2 = \frac{|S_{21}|^2}{(1 - |S_{11}|^2)(1 - |S_{22}|^2)} \quad (40)$$

An amplifier is said to be unconditionally stable if the real parts of its input and output impedances remain positive when the load and source impedances, respectively, are changed arbitrarily, but keeping their real parts positive. Let us next consider the condition for an amplifier to be unconditionally stable. For the input impedance, we require $|S_{11}'|$ to be less than 1 when r_2 is changed arbitrarily, but keeping $|r_2| < 1$. Similarly, for the output impedance, $|S_{22}'| < 1$ is required when $|r_1| < 1$. Using (32), a little manipulation shows that the necessary and sufficient conditions are given by

$$\begin{aligned} |S_{12}S_{21}| &< 1 - |S_{11}|^2 \\ |S_{12}S_{21}| &< 1 - |S_{22}|^2 \\ 2|S_{12}S_{21}| &< 1 + |S_{12}S_{21} - S_{11}S_{22}|^2 - |S_{11}|^2 - |S_{22}|^2 \end{aligned} \quad (41)$$

The last condition is identical with that under which simultaneous matching is possible when $|S_{12}S_{21}| \neq 0$. It is interesting to note that simultaneous matching is possible for any amplifier which is unconditionally stable, but the reverse is not necessarily true.

From the first two conditions in (41), it can be shown that B, as given by (39), is negative when the amplifier is unconditionally stable. Therefore, the lower sign applies in this case on the right-hand side of (37). Furthermore, since $|S_{12}|/|S_{21}|$ is invariant to changes in source and load impedances, as can be shown from (32), and $|S_{21}'|^2$ in (37) is similarly invariant (from physical reasoning), k is also invariant to changes in source and load impedances.

VIII. Choice of Phase

The phase of the incident wave a_i is equal to that of the open circuit voltage E of the ith circuit and the phase of the reflected wave b_i is that of $E - 2\{\text{Re } Z_i\} I_i$. When the ith circuit has no source, b_i has the phase of the voltage across the resistance in the series representation of the circuit. However, since it is only the square of the magnitude of the waves that we have used for the power discussion in Section II, an arbitrary phase could be assigned to each wave without changing the power relation. Thus, in place of (1), we could define the waves a_i and b_i by

$$a_i = \frac{V_i + Z_i I_i}{2\sqrt{|\text{Re } Z_i|}} e^{j\phi_i}, \qquad b_i = \frac{V_i - Z_i^* I_i}{2\sqrt{|\text{Re } Z_i|}} e^{j\psi_i} \quad (42)$$

respectively, where ϕ_i and ψ_i are arbitrary angles. The scattering matrix S in this case is defined through the relation

$$b = Sa$$

where a and b are the vectors with their ith components being a_i and b_i given by (42). The reciprocal condition (21) is replaced by

$$S_t = (MN)^{-1} PSPMN$$

where M and N are the diagonal matrices whose ith diagonal components are $e^{j\phi_i}$ and $e^{j\psi_i}$, respectively. The lossless condition (24) remains the same. However, the equation corresponding to (30) has two additional terms

$$2S^+PS \frac{\partial M}{\partial \omega} M^{-1} + 2S^+PN \frac{\partial N^{-1}}{\partial \omega} S,$$

in the bracket of the left-hand side of (30). The original form used by Penfield [3] is just a special case of the above definition. The phases were chosen so that, for $\text{Re } Z_i > 0$, $e^{j\phi_i} = e^{j\psi_i} = 1$ and for $\text{Re } Z_i < 0$, $e^{j\phi_i} = e^{j\psi_i} = -j$. In this case MN is equal to P and the reciprocal relation takes a simple form: $S_t = S$.

Another interesting choice of the phases is given by

$$e^{j\phi_i} = \sqrt{\frac{|Z_i|}{Z_i}}, \qquad e^{j\psi_i} = \sqrt{\frac{|Z_i|}{-Z_i^*}}$$

where

$$-\frac{\pi}{2} \leq \phi_i \leq \frac{\pi}{2}, \qquad -\frac{\pi}{2} \leq \psi_i \leq \frac{\pi}{2}$$

The significance of this choice lies in the following fact. When we replace every quantity appearing in the definition of the waves (42) by the corresponding dual quantity, the waves stay the same. Thus

$$a_i = \frac{V_i + Z_i I_i}{2\sqrt{|\text{Re } Z_i|}} \sqrt{\frac{|Z_i|}{Z_i}} = \frac{I_i + Y_i V_i}{2\sqrt{|\text{Re } Y_i|}} \sqrt{\frac{|Y_i|}{Y_i}}$$

$$b_i = \frac{V_i - Z_i^* I_i}{2\sqrt{|\text{Re } Z_i|}} \sqrt{\frac{|Z_i|}{-Z_i^*}} = \frac{I_i - Y_i^* V_i}{2\sqrt{|\text{Re } Y_i|}} \sqrt{\frac{|Y_i|}{-Y_i^*}}$$

The traveling waves defined by (43) in Section IX have this property. However, the power waves defined by (1) have not.

IX. Comparison with Traveling Waves

The traveling waves along a transmission line can be defined by

$$a(z) = \frac{V(z) + Z_0 I(z)}{2\sqrt{Z_0}}, \qquad b(z) = \frac{V(z) - Z_0 I(z)}{2\sqrt{Z_0}} \quad (43)$$

where $V(z)$ and $I(z)$ are the voltage and current at a point z along the line and Z_0 is the characteristic impedance. If we consider Z_0 with a positive real value, the expression for the power waves becomes identical with that for the traveling waves. Therefore, all the conditions which the scattering matrix for traveling waves must satisfy in order to represent certain networks can be obtained if we set P equal to a unit matrix I in the corresponding conditions for the power wave scattering matrix. Thus, the lossless condition becomes $S^+S=I$, the reciprocal condition $S_t=S$ and (35), (36), and (41) stay the same. However, the interpretation of these results is different. For example, let us consider the condition $|S_{ij}|^2=|S_{ji}|^2$ for a reciprocal network. Assuming that all the characteristic impedances of the lines are real and positive, the direct interpretation of this condition is as follows. The power coming out from the jth port when the incoming power into the ith port is unity is equal to the power coming out from the ith port when the incoming power into the jth port is unity, provided that all the circuits connected to the far ends of the lines are matched to the line characteristic impedances. However, the last restriction is generally too stringent for practical applications. And it is only after a little manipulation that we discover the power reciprocal relation given in Section IV. Thus, according to the particular problem we have, a choice must be made between the traveling wave and power wave representation. For instance, if we want to discuss the properties of a junction irrespective of the impedances connected to the terminals, the traveling waves may be more convenient. On the other hand, for the power relation between circuits connected through a junction, the power wave representation is more suitable. One may ask then what is the relation between the traveling waves and the power waves. When Z_0 is real and positive, there is no difference in the expressions of the power waves and the traveling waves. Therefore, in this case, the net power in the z direction is given by $|a(z)|^2-|b(z)|^2$. However, when Z_0 is complex, the situation is different. $|a(z)|^2-|b(z)|^2$ is calculated to be Re $\{Z_0V^*I\}/|Z_0|$, which is not equal to the power Re $\{VI^*\}$. Thus, each traveling wave cannot be considered to bring the power expressed by the square of the magnitude. Further, since the traveling wave reflection coefficient is given by $(Z_L-Z_0)/(Z_L+Z_0)$ and the maximum power transfer takes place when $Z_L=Z_0^*$, where Z_L is the load impedance, it is only when there is a certain reflection in terms of traveling waves that the maximum power is transferred from the line to the load. Thus, we have seen that, in general, the traveling wave concept is not so closely related with the power.

X. Conclusion

The physical meaning of power waves and the properties of the scattering matrix are presented. Although the power waves are the result of just one of an infinite number of possible linear transformations of voltage and current, it has been shown that, for certain applications, they give a clearer and more straightforward understanding of the power relations between circuit elements connected through a multiport network.

Appendix I

Let us prove the reciprocal condition (21). Using (18), (20), and the obvious relations $F_t=F$, $G_t=G$, the left-hand side of (21) can be rewritten in the form

$$\begin{aligned} S_t &= F_t^{-1}(Z+G)_t^{-1}(Z-G^+)_tF_t \\ &= F_t^{-1}(Z_t+G_t)^{-1}(Z_t-G_t^+)F_t \\ &= F^{-1}(Z+G)^{-1}(Z-G^+)F \end{aligned}$$

We wish to prove that this last expression is equal to the right-hand side of (21), which is given by

$$PSP = PF(Z-G^+)(Z+G)^{-1}F^{-1}P$$

To do so, since $P=P^{-1}$, we have only to prove the following equation.

$$(Z-G^+)FPF(Z+G) = (Z+G)FPF(Z-G^+).$$

Performing the matrix product, the above equation becomes

$$\begin{aligned} &ZFPFZ + ZFPFG - G^+FPFZ - G^+FPFG \\ &\quad = ZFPFZ - ZFPFG^+ + GFPFZ - GFPFG^+, \end{aligned}$$

of which the first terms in both sides are the same and the last terms are equal to each other. Thus, all that we have to prove is

$$ZFPFG - G^+FPFZ = -ZFPFG^+ + GFPFZ$$

or

$$ZFPF(G+G^+) = (G^++G)FPFZ \quad (44)$$

Since

$$\begin{aligned} FPF(G+G^+) &= \tfrac{1}{2}I \\ (G^++G)FPF &= \tfrac{1}{2}I, \end{aligned}$$

the validity of (44) is obvious. This completes the proof of (21).

Appendix II

The derivation of (30) will be given briefly. From Maxwell's equations and an appropriate definition of voltage and current at the reference planes, after a little manipulation, we have

$$i^+ \frac{\partial v}{\partial \omega} + v^+ \frac{\partial i}{\partial \omega} = j \int (\mu H^* \cdot H + \epsilon E^* \cdot E) dv \qquad (45)$$

from which (29) is derived. The expressions for v and i in terms of a and b are

$$v = 2PF(G^+ a + Gb), \qquad i = 2FP(a - b)$$

Substituting these expressions, the left-hand side of (45) becomes

$$\begin{aligned} i^+ \frac{\partial v}{\partial \omega} + v^+ \frac{\partial i}{\partial \omega} = 4 \Bigg\{ & a^+ \left(F^+ \frac{\partial FG^+}{\partial \omega} + GF^+ \frac{\partial F}{\partial \omega} \right) a \\ & + a^+ (F^+ F G^+ + G F^+ F) \frac{\partial a}{\partial \omega} \\ & - b^+ \left(F^+ \frac{\partial FG}{\partial \omega} + G^+ F^+ \frac{\partial F}{\partial \omega} \right) b \\ & - b^+ (F^+ F G + G^+ F^+ F) \frac{\partial b}{\partial \omega} \Bigg\} \end{aligned} \qquad (46)$$

Since

$$F^+ F G^+ + G F^+ F = \tfrac{1}{2} P$$
$$F^+ F G + G^+ F^+ F = \tfrac{1}{2} P$$
$$F^+ \frac{\partial FG^+}{\partial \omega} + GF^+ \frac{\partial F}{\partial \omega} = \tfrac{1}{4} K$$
$$F^+ \frac{\partial FG}{\partial \omega} + G^+ F^+ \frac{\partial F}{\partial \omega} = -\tfrac{1}{4} K$$

the right-hand side of (46) reduces to

$$a^+ K a + b^+ K b + 2 \left(a^+ P \frac{\partial a}{\partial \omega} - b^+ P \frac{\partial b}{\partial \omega} \right)$$

Using $b = Sa$, this can be rewritten in the form

$$a^+ K a + a^+ S^+ K S a + 2 \left(a^+ P \frac{\partial a}{\partial \omega} - a^+ S^+ P \frac{\partial S}{\partial \omega} a - a^+ S^+ P S \frac{\partial a}{\partial \omega} \right)$$

Because of (24), the first and last terms in the bracket cancel each other. Therefore, (45) becomes

$$a^+ \left(K + S^+ K S - 2 S^+ P \frac{\partial S}{\partial \omega} \right) a = j \int (\mu H^* \cdot H + \epsilon E^* \cdot E) dv$$

which is equivalent to (30).

Appendix II

An outline of the derivation of (32) will be given in this appendix. From (18),

$$S' = F'(Z - G'^+)(Z + G')^{-1} F'^{-1} \qquad (47)$$

where F' and G' represent F and G, respectively, when Z_i is replaced by Z_i' everywhere. Substituting (19) into (47) and using Γ, defined by

$$\Gamma = (G' - G)(G' + G^+)^{-1}, \qquad (48)$$

S' can be rewritten in the form

$$F' F^{-1} (I - S)^{-1} (S - \Gamma^+)(I - \Gamma^+)(I - \Gamma) \cdot (I - S\Gamma)^{-1} (I - S) F F'^{-1}$$

Since

$$(I - S)^{-1}(S - \Gamma^+)(I - \Gamma^+)^{-1} = (I - \Gamma^+)^{-1}(S - \Gamma^+)(I - S)^{-1}$$

$$(I - \Gamma)(I - S\Gamma)^{-1}(I - S) = (I - S)(I - \Gamma S)^{-1}(I - \Gamma)$$

S' becomes

$$S' = A^{-1}(S - \Gamma^+)(I - \Gamma S)^{-1} A^+$$

where A is a diagonal matrix defined by

$$A = F'^{-1} F (I - \Gamma^+).$$

Calculation of the ith diagonal component A_i shows that

$$A_i = \frac{1 - r_i^*}{|1 - r_i|} \sqrt{|1 - r_i r_i^*|}$$

where r_i is the ith diagonal component of Γ and (referring (48)) is given by

$$r_i = \frac{Z_i' - Z_i}{Z_i' + Z_i^*}.$$

From this, r_i is interpreted as the power wave reflection coefficient of Z_i' with respect to Z_i^*.

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