Chapter 6: Microwave Resonators

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Microwave Resonators Applications

Microwave resonators are used in a variety of applications such as:
- Filters
- Oscillators
- Frequency meters
- Tuned Amplifier

The operation of microwave resonators are very similar to that of the lumped-element resonators of circuit theory, thus we will review the basic of series and parallel RLC resonant circuits first. We will derive some of the basic properties of such circuits.
Series Resonant Circuit

Near the resonance frequency, a microwave resonator can be modeled as a series or parallel RLC lumped-element equivalent circuit.

\[ Z_{in} = R + j\omega L - j\frac{1}{\omega C} \]

\[ P_m = \frac{1}{2}|I|^2 \quad Z_{in} = \frac{1}{2}|I|^2 \left( R + j\omega L - j\frac{1}{\omega C} \right) \]

Average magnetic energy: \[ W_m = \frac{1}{4}|I|^2 \frac{1}{L} \]

Average electric energy: \[ W_e = \frac{1}{4}|I|^2 \frac{1}{\omega^2 C} \]

Resonance occurs when the average stored magnetic \( (W_m) \) and electric energies \( (W_e) \) are equal and \( Z_{in} \) is purely real.

A series RLC resonator and its response. (a) The series RLC circuit. (b) The input impedance magnitude versus frequency.
Series Resonators

- The frequency in which \( \omega_o = \frac{1}{\sqrt{LC}} \)
  is called the **resonant frequency**.
- Another important factor is the **Quality Factor** \( Q \).

\[
Q = \frac{\text{Average Energy Stored}}{\text{Energy Loss / Second}}
\]

\[
BW = \frac{1}{Q} \quad Q = \frac{\omega_o L}{R} = \frac{1}{\omega_o RC}
\]

Near the resonance \( \omega = \omega_o + \Delta \omega \)

\[
Z_{in} = R + j\omega L(1 - \frac{1}{\omega^2 LC}) = R + j\omega L(\frac{\omega^2 - \omega_o^2}{\omega^2})
\]

\[
= R + j2RQ\Delta \omega \quad \frac{\omega}{\omega_o}
\]

\[
\omega^2 - \omega_o^2 = (\omega - \omega_o)(\omega + \omega_o) \approx 2\omega \Delta \omega
\]

\[
Z_{in} = R + j2L\Delta \omega
\]
Series Resonators

The input impedance magnitude versus frequency.

\[ Q = \frac{\omega_o L}{R} = \frac{1}{\omega_o RC} \]

Fractional Bandwidth is defined as:
\[ \frac{\Delta \omega}{\omega_o} = \frac{BW}{2} \]
\[ BW = \frac{1}{Q} \]

and happens when the average (real) power delivered to the circuit is one-half that delivered at the resonance.

- Bandwidth increases as R increases.
- Narrower bandwidth can be achieved at higher quality factor (Smaller R).
Parallel Resonators

A parallel RLC resonator and its response. (a) The parallel RLC circuit. (b) The input impedance magnitude versus frequency.

\[
Z_{in} = \left( \frac{1}{R} + \frac{1}{j\omega L} + j\omega C \right)^{-1}
\]

Resonance occurs when the average stored magnetic and electric energies are equal and \( Z_{in} \) is purely real. The input impedance at resonance is equal to \( R \).

\[
Q = \frac{\text{Average Energy Stored}}{\text{Energy Loss / Second}}
\]

\[
\omega_o = \frac{1}{\sqrt{LC}} \quad Q = \frac{R}{\omega_o L} = \omega_o RC
\]

Note: Resonance frequency is equal to the series resonator case. \( Q \) is inversed.
Parallel Resonators

The input impedance magnitude versus frequency.

\[ Q = \frac{R}{\omega_0 L} = \omega_o RC \quad BW = 1/Q \]

- Bandwidth reduces as R increases.
- Narrower bandwidth can be achieved at higher quality factor (Larger R).

Close to the resonance frequency:

\[ \omega = \omega_o + \Delta \omega \]

\[ Z_{in} \approx \left( \frac{1}{R} + 2j\Delta \omega C \right)^{-1} = \frac{R}{1 + 2jQ\Delta \omega / \omega_o} \]

As \( R \to \infty \),

\[ Z_{in} = \frac{1}{j2C(\omega - \omega_o)} \]
Loaded and Unloaded Q Factor

The Q factors that we have calculated were based on the characteristic of the resonant circuit itself, in the absence of any loading effect (Unloaded Q).

In practice a resonance circuit is always connected to another circuitry, which will always have the effect of lowering the overall Q (Loaded Q).

\[ Q_e = \begin{cases} \frac{\omega_o L}{R_L} & \text{for series connection} \\ \frac{R_L}{\omega_o L} & \text{for parallel connection} \end{cases} \]
Loaded Q Factor

If the resonator is a series RLC and coupled to an external load resistor $R_L$, the effective resistance is:

$$ R_e = R + R_L $$

If the resonator is a parallel RLC and coupled to an external load resistor $R_L$, the effective resistance is:

$$ R_e = R \cdot R_L / (R + R_L) $$

Then the loaded Q can be written as:

$$ \frac{1}{Q_L} = \frac{1}{Q} + \frac{1}{Q_e} $$

**Note:** Loaded Q factor is always smaller than Unloaded Q.
### TABLE 6.1  Summary of Results for Series and Parallel Resonators

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Series Resonator</th>
<th>Parallel Resonator</th>
</tr>
</thead>
<tbody>
<tr>
<td>Input Impedance/admittance</td>
<td>$Z_{in} = R + j\omega L - j\frac{1}{\omega C}$</td>
<td>$Y_{in} = \frac{1}{R} + j\omega C - j\frac{1}{\omega L}$</td>
</tr>
<tr>
<td></td>
<td>$\simeq R + j\frac{2Q\Delta\omega}{\omega_0}$</td>
<td>$\simeq \frac{1}{R} + j\frac{2Q\Delta\omega}{R\omega_0}$</td>
</tr>
<tr>
<td>Power loss</td>
<td>$P_{loss} = \frac{1}{2}</td>
<td>I</td>
</tr>
<tr>
<td>Stored magnetic energy</td>
<td>$W_m = \frac{1}{4}</td>
<td>I</td>
</tr>
<tr>
<td>Stored electric energy</td>
<td>$W_e = \frac{1}{4}</td>
<td>I</td>
</tr>
<tr>
<td>Resonant frequency</td>
<td>$\omega_0 = \frac{1}{\sqrt{LC}}$</td>
<td>$\omega_0 = \frac{1}{\sqrt{LC}}$</td>
</tr>
<tr>
<td>Unloaded $Q$</td>
<td>$Q = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 RC}$</td>
<td>$Q = \omega_0 RC = \frac{R}{\omega_0 L}$</td>
</tr>
<tr>
<td>External $Q$</td>
<td>$Q_e = \frac{\omega_0 L}{R_L}$</td>
<td>$Q_e = \frac{R_L}{\omega_0 L}$</td>
</tr>
</tbody>
</table>
Transmission Line Resonators (Short Circuited)

- Ideal lumped element (R, L and C) are usually impossible to find at microwave frequencies.
- We can design resonators with transmission line sections with different lengths and terminations (Open or Short).
- Since we are interested in the Q of these resonators we will consider the Lossy Transmission Line.

For the special case of: \( \ell = \lambda / 2 \)

\[
Z_{in} = Z_o \tanh(\alpha + j\beta)\ell
\]

\[
Z_{in} = Z_o \frac{\tanh \alpha \ell + j \tan \beta \ell}{1 + j \tanh \beta \ell \cdot \tanh \alpha \ell}
\]

\[
Q = \frac{\omega_o L}{R} = \frac{\beta}{2\alpha}
\]
Transmission Line Resonators (Short Circuited)

- The resonance occurs for \( \ell = \frac{n\lambda}{2} \) \( n = 1, 2, 3, \ldots \)

\[
Z_{in} = Z_o[\alpha \ell + j(\Delta \omega \pi / \omega_o)]
\]

\[
Q = \frac{\omega_o L}{R} = \frac{\beta}{2\alpha}
\]

A short-circuited length of lossy transmission line and the voltage distributions for \( n=1 \), \( \ell = \frac{\lambda}{2} \).
An open-circuited length of lossy transmission line, and the voltage distributions for $n = 1$ resonators.
A rectangular resonant cavity, and the electric field distributions for the TE_{101} and TE_{102} resonant modes.

Rectangular waveguide cavity resonator

We can look at them as short circuit section of transmission line

\[ \beta_{mn} = \sqrt{\omega^2 \mu \varepsilon - \left( \frac{m\pi}{a} \right)^2 - \left( \frac{n\pi}{b} \right)^2} \]

Boundary conditions enforced

\[ E_x = E_y = 0 \text{ for } z = 0, d \]

\[ \Rightarrow \beta_{mn} d = l\pi \Rightarrow \beta_{mn} = \frac{l\pi}{d} \quad l = 1, 2, 3, \ldots \]

\[ \Rightarrow \left( \frac{l\pi}{d} \right)^2 = \omega^2 \mu \varepsilon - \left( \frac{m\pi}{a} \right)^2 - \left( \frac{n\pi}{b} \right)^2 \]

\[ \frac{2\pi f}{c} = \sqrt{\left( \frac{l\pi}{d} \right)^2 + \left( \frac{m\pi}{a} \right)^2 + \left( \frac{n\pi}{b} \right)^2} \]

then the resonance frequency

\[ f = \frac{c}{2\pi} \sqrt{\left( \frac{l}{d} \right)^2 + \left( \frac{m}{a} \right)^2 + \left( \frac{n}{b} \right)^2} \]

if \( b < a < d \) the lowest and dominant resonant TE (resp. TM) mode will be TE_{101} (resp. TM_{110})
Rectangular waveguide cavity resonator
Q factor for TE10

\[
Q = \left( \frac{1}{Q_c} + \frac{1}{Q_d} \right)^{-1}
\]

\[
Q_d = \frac{1}{\tan \delta}
\]

\[
Q_c = \frac{(2\pi ad/\lambda)^3 \eta}{2\pi^2 R_s} \frac{1}{\left(2d^2 a^3 b + 2bd^3 + l^2 a^3 d + ad^3 \right)}
\]

\[
R_s = \sqrt{\frac{\omega \mu_0}{2\sigma}}
\]

\[
\eta = \sqrt{\frac{\mu}{\varepsilon}}
\]
Resonators - coupling

Critical coupling occurs when \( 2Q_L = Q_e = Q_u \)

If we define coupling coefficient \( g = \frac{Q_u}{Q_e} \) then we have

- undercoupled resonator if \( g < 1 \)
- critically coupled resonator if \( g = 1 \) (resonator matched to the feed line)
- overcoupled resonator if \( g > 1 \)

Series RLC circuit

\[
\begin{align*}
\text{series RLC} & \quad g = \frac{Z_0}{R} \\
\text{parallel RLC} & \quad g = \frac{R}{Z_0}
\end{align*}
\]
- Critical coupling $Z_{in}(\omega_0) = Z_0$
Gap coupled microstrip resonator resonators

\[ \frac{\lambda}{2} \]

Serial RLC coupling capacitor acts as impedance inverter

\[ z = \frac{Z}{Z_o} = -j \left( \frac{\tan \beta l + b_c}{b_c \tan \beta l} \right) \quad \text{where} \quad b_c = Z_o \omega C \]

Resonance occurs when \( \tan \beta l + b_c = 0 \)

The coupling of the feed line to the resonator lowers its resonant frequency
Gap coupled microstrip resonator resonators

\[ Z(\omega) = Z_o \left( \frac{\pi}{2Qb_c^2} + j \frac{\pi(\omega - \omega_o)}{\omega_o b_c^2} \right) \]

\[ \Rightarrow R = Z_o \left( \frac{\pi}{2Qb_c^2} \right) \Rightarrow b_c = \sqrt{\frac{\pi}{2Q}} \]

the coupling coefficient \( g = \frac{Z_o}{R} = \frac{\pi}{2Q_u b_c^2} \)

for critical coupling \( Z_o = R \ \Rightarrow b_c = \sqrt{\frac{\pi}{2Q}} \)

for undercoupled resonator \( \Rightarrow b_c < \sqrt{\frac{\pi}{2Q}} \)

for overcoupled resonator \( \Rightarrow b_c > \sqrt{\frac{\pi}{2Q}} \)
Determine $Q_U$ from 2-port measurement

\[ Z_0 \quad R \quad L \quad C \quad Z_0 \]

\[ Z_0 \quad Z_0 \]

@ $f_o$:

\[ [Z] = \begin{bmatrix} R & R \\ R & R \end{bmatrix}, \quad [S] = \begin{bmatrix} \frac{R}{R + 2Z_0} & \frac{2Z_0}{R + 2Z_0} \\ \frac{2Z_0}{R + 2Z_0} & \frac{R}{R + 2Z_0} \end{bmatrix} \]

... prob 4.11

\[ Q_U = \frac{\omega_o L}{R}, \quad Q_e = \frac{\omega_o L}{R_L} = \frac{\omega_o L}{2Z_0} \Rightarrow g = \frac{Q_U}{Q_e} = \frac{2Z_0}{R} \]

\[ S_{21}(f_o) = \frac{2Z_0}{R + 2Z_0} = \frac{g}{1 + g} \Rightarrow g = \frac{S_{21}(f_o)}{1 - S_{21}(f_o)} \]

\[ \frac{1}{Q_L} = \frac{1}{Q_e} + \frac{1}{Q_U} = \frac{1}{Q_U} (1 + g), \text{ measure } Q_L = \frac{f_o}{\Delta f_{3dB}} \Rightarrow Q_U = Q_L (1 + g) \]
Solved problems: Prob. 6.22 A parallel resonator, calculate \( C_0 \) for critical coupling and \( f_r \).

\[ Y_m(\omega) \approx \frac{1}{R} + j \frac{2Q_0 \Delta \omega}{R \omega_0} = 10^{-3} + j \times 50.5 \times 10^{-3} \frac{\Delta \omega}{\omega_0}, Q_U = \frac{R}{\omega_0 L} = 25.3, \omega_0 = \frac{1}{\sqrt{LC}} = 31.4 \times 10^9 \]

\[ y_m = 50 \times Y_m = 0.05 + j 2.53 \frac{\Delta \omega}{\omega_0} = \frac{1}{1 + ja} = \frac{1}{1 + a^2} - j \frac{a}{1 + a^2} \rightarrow 0.05 = \frac{1}{1 + a^2} \rightarrow a = 4.36 \]

\[ \therefore z_m + \frac{1}{\frac{j \omega C_0 Z_0}{1 + a^2}} = 1 \rightarrow z_m = 1 + j \frac{1}{\omega C_0 Z_0} = 1 + ja \]

\[ -\frac{a}{1 + a^2} = -\frac{4.36}{20} = 2.53 \frac{\Delta \omega}{\omega_0} \rightarrow \frac{\Delta \omega}{\omega_0} = -0.086 \]

\[ \rightarrow f_r = f_o - 0.086 f_o = 4.57 \text{GHz} \]

\[ \frac{1}{\omega_r C_0 Z_0} = 4.36 \rightarrow C_0 = 0.16 \text{pF} \]
Dielectric resonators

\[10 < \varepsilon_r < 100, \quad \text{TE}_{01\delta} \text{ mode} \quad \delta = \frac{2L}{\lambda_g} < 1, \quad Q_d \approx \frac{1}{\tan \delta}\]

Ex 6.5 \( \varepsilon_r = 95, \tan \delta = 0.001, \ a = 0.413 \text{ cm} \)

\[\rightarrow f = 3.4 \text{GHz}, \quad Q_d = 1000\]
1. Q = 100 인 질량 RLC 공진회로
   a) 설계: 공진 주파수 → 전계자가 결정, 오차 계산
   설계자가 결정

   b) QuickSmith로 스미스도트맵 원리 검토
      제작도시

2. RLC 병렬공진회로에 대해 반복

3. 1의 회로의 우라주에 따른 입력 임시적인 계산 Python 프로그램 작성

4. 2에 대해 3항 반복