# Lecture 11 Maxwell's Equations

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# 1. Maxwell's Equations

- Maxwell's equations (ME)

$$\nabla \cdot \mathbf{D} = \rho \quad \text{(Gauss's law)}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \text{(Faraday's law)}$$

$$\nabla \cdot \mathbf{B} = 0 \quad \text{(Gauss's law)}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \quad \text{(Ampere's law)}$$

- ME in terms of **E** and **H**

$$\nabla \cdot \mathbf{E} = \rho / \varepsilon$$

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} \quad \text{(Faraday's law)}$$

$$\nabla \cdot \mathbf{H} = 0 \quad \text{(Gauss's law)}$$

$$\nabla \times \mathbf{H} = \sigma \mathbf{E} + \varepsilon \frac{\partial \mathbf{E}}{\partial t} \quad \text{(Ampere's law)}$$

- Materials electrical properties

$$\mathbf{D} = \varepsilon \mathbf{E} \quad \text{(dielectric property)}$$

$$\mathbf{B} = \mu \mathbf{H} \quad \text{(magnetic property)}$$

$$\mathbf{J} = \sigma \mathbf{E} \quad \text{(conductive property)}$$

- Sinusoidal field

$$\mathcal{E} = \mathbf{E}_0 \cos(\omega t + \phi_e)$$

$$\mathcal{H} = \mathbf{H}_0 \cos(\omega t + \phi_h)$$

$$\frac{\partial}{\partial t} \rightarrow j\omega \quad \text{and} \quad \int dt \rightarrow \frac{1}{j\omega}$$

$\mathbf{E} = \mathbf{E}_0 e^{j\phi_e}$ : electric field phasor

$\mathbf{H} = \mathbf{H}_0 e^{j\phi_h}$ : magnetic field phasor

- ME in phasor form

$$\nabla \cdot \mathbf{E} = \rho / \varepsilon$$

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H}$$

$$\nabla \cdot \mathbf{H} = 0$$

$$\nabla \times \mathbf{H} = (\sigma + j\omega\varepsilon)\mathbf{E}$$

# 2. Power and Energy in Electromagnetic Fields

- Power in terms of voltage and current

DC:

$$W = \frac{1}{2}VQ = \frac{1}{2}\frac{Q^2}{C} \quad \text{(dc capacitor stored energy)}$$

$$P = \frac{dW}{dt} = \frac{1}{2}\frac{2Q}{C}\frac{dQ}{dt} = VI \quad \text{(dc capacitor power)}$$

AC:

$$v(t) = V_0 \cos(\omega t + \phi_v), \; i(t) = I_0 \cos(\omega t + \phi_i)$$

$$V = V_0 e^{j\phi_v}, \; I = I_0 e^{j\phi_i}$$

$$p(t) = vi = \frac{1}{2}V_0 I_0 \left[\cos(\phi_v - \phi_i) + \cos(2\omega t + \phi_v + \phi_i)\right]$$

$$P_{\text{dc}} = \frac{1}{2}V_0 I_0$$

$$P_{\text{ac}} = \frac{1}{2}V_0 I_0 \cos(2\omega t + \phi_v + \phi_i)$$

$$P_{\text{av}} = < p > = \frac{1}{T}\int_0^T p(\tau)d\tau = \frac{1}{2}V_0 I_0 \cos(\phi_v - \phi_i)$$

$$P_{\text{av}} = \frac{1}{2}\text{Re}(VI^*) = \frac{1}{2}\text{Re}(V^* I)$$

$$P = \frac{1}{2}VI^* = \frac{1}{2}V^* I \quad \text{(complex power)}$$

- Power loss in dielectric materials

$$w = \frac{1}{2} vq = \frac{1}{2} \frac{q^2}{C} \quad \text{(capacitor stored energy)}$$

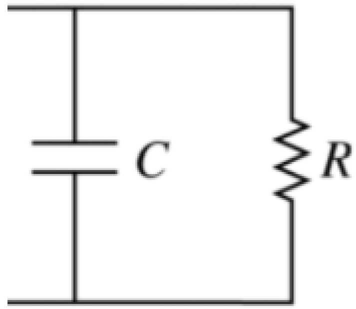
$$p = vi,\ i = C\frac{dv}{dt} \rightarrow P = \frac{1}{2} V^* I = \frac{1}{2} V^* (j\omega C V) = \frac{1}{2} (j\omega) C |V|^2$$

$$C = (\varepsilon' - j\varepsilon'') \frac{S}{d}$$

$$P = \frac{1}{2} (\omega\varepsilon'') \frac{S}{d} |V|^2 + j\frac{1}{2} (\omega\varepsilon') \frac{S}{d} |V|^2$$

$$P = \frac{1}{2} \frac{|V|^2}{R} + j\omega \frac{1}{2} C |V|^2 = \frac{1}{2} Y |V|^2$$

$$Y = \frac{1}{R} + j\omega C$$




$$R = \frac{d}{(\omega\varepsilon'')S} = \frac{d}{\sigma S} \quad \text{(resistance in parallel with capacitance)}$$

$$C = \varepsilon' \frac{S}{d} \quad \text{(capacitance)}$$

- Complex permittivity and electric loss tangent

$\varepsilon = \varepsilon' - j\varepsilon''$ (complex permittivity)

$\tan\delta = \dfrac{\varepsilon''}{\varepsilon'}$ (loss tangent)

$\varepsilon' = \varepsilon_0 \varepsilon_r$: accounts for dielectric polarization

$\varepsilon''$ : accounts for dielectric loss

$\varepsilon_0 = 8.854 \times 10^{-12}$ F/m (permittivity of vacuum)

$\varepsilon_r$ : relative permittivity = dielectric constant

$\varepsilon', \varepsilon''$: materials with dielectric polarization and dielectric loss

$\varepsilon', \sigma = \omega\varepsilon''$ : materials with dielectric polarization and conduction loss

- Power loss in magnetic materials

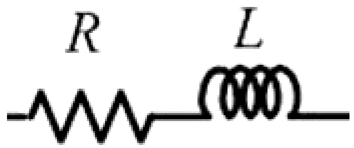
$$w = \frac{1}{2} i\phi = \frac{1}{2} L i^2 \quad \text{(magnetic energy stored in an inductor)}$$

$$p = vi,\ v = L\frac{di}{dt} \rightarrow P = \frac{1}{2} I^* V = \frac{1}{2} I^* (j\omega L I) = \frac{1}{2}(j\omega) L |I|^2$$

$$L = (\mu' - j\mu'')\frac{S}{d} \quad \text{(inductance of a torroidal inductor)}$$

$$P = \frac{1}{2}(\omega\mu'')\frac{S}{d}|I|^2 + j\frac{1}{2}(\omega\mu')\frac{S}{d}|I|^2$$

$$P = \frac{1}{2} R |I|^2 + j\omega\frac{1}{2} L |I|^2 = \frac{1}{2} Z |V|^2$$




$$Z = R + j\omega L$$

$$R = \omega\mu''\frac{S}{d} \quad \text{(resistor in series with a lossless inductor)}$$

$$L = \mu'\frac{S}{d} \quad \text{(inductance)}$$

- Complex permeability and magnetic loss tangent

$\mu = \mu' - j\mu''$ (complex permeability)

$\tan\delta_m = \frac{\mu''}{\mu'}$ (magnetic loss tangent)

$\mu' = \mu_0\mu_r$: accounts for magnetization of a material

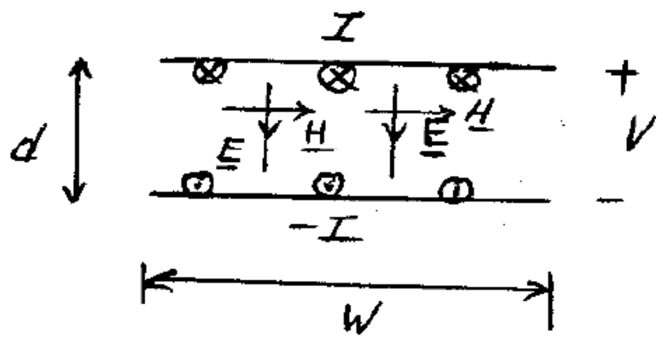
$\mu''$: accounts for magnetic loss of a material

$\mu_0 = 4\pi \times 10^{-7}$ H/m (permeability of vacuum)

$\mu_r$: relative permeability

- Power density of electromagnetic fields

- Use an ideal parallel-plate capacitor to understand the concept




$$V = Ed$$

$$I = WH$$

$$P = \frac{1}{2} I^* V = \frac{1}{2} V^* I$$

$$P = \frac{1}{2} H^* E(Wd) = \frac{1}{2} E^* H(Wd)$$

$$S \equiv \frac{1}{2} H^* E = \frac{1}{2} E^* H$$ (complex power density of electromagnetic field)

$$\mathbf{S} = \frac{1}{2} \mathbf{E}^* \times \mathbf{H} = \frac{1}{2} \mathbf{E} \times \mathbf{H}^*$$ (Poynting vector = electromagnetic power density)

$\mathbf{S}$: complex power density vector (in the direction of wave propagation)

- Stored energy in electric field

$$w = \frac{1}{2}Cv^2 = \frac{1}{2}\varepsilon\frac{S}{d}(Ed)^2 = \frac{1}{2}\varepsilon E^2(Sd): \text{ parallel-plate capacitor}$$

$$w = \frac{1}{2}\mathbf{D}\cdot\mathbf{E} = \frac{1}{2}DE \text{ (J/m}^3\text{, electric field energy density)}$$

$$W = \frac{1}{4}DE^* = \frac{1}{4}E^*(\varepsilon' - j\varepsilon'')E = \frac{1}{4}(\varepsilon' - j\varepsilon'')|E|^2 \text{ (phasor for electric energy density)}$$

$$P = (j2\omega)W = \frac{1}{2}(\omega\varepsilon'')|E|^2 + j\frac{1}{2}(\omega\varepsilon')|E|^2 \text{ (phasor for electric power density)}$$

$$P_r = \frac{1}{2}(\omega\varepsilon'')|E|^2 = \frac{1}{2}\sigma|E|^2 = \frac{1}{2}JE \text{ (electric loss power density)}$$

$$P_i = \frac{1}{2}(\omega\varepsilon')|E|^2 \text{ (stored electric power density)}$$

- Stored energy in magnetic field

$$w = \frac{1}{2}Li^2 = \frac{1}{2}\mu\frac{S}{d}(Hd)^2 = \frac{1}{2}\mu E^2(Sd):\ \text{torroidal inductor}$$

$$w = \frac{1}{2}\mathbf{B}\cdot\mathbf{H} = \frac{1}{2}BH\ \text{(J/m}^3\text{, magnectic field energy density)}$$

$$W = \frac{1}{4}BE^* = \frac{1}{4}H^*(\mu' - j\mu'')E = \frac{1}{4}(\mu' - j\mu'')|H|^2\ \text{(phasor for magnetic energy density)}$$

$$P = (j2\omega)W = \frac{1}{2}(\omega\mu'')|H|^2 + j\frac{1}{2}(\omega\mu')|H|^2\ \text{(phasor for magnetic power density)}$$

$$P_r = \frac{1}{2}(\omega\mu'')|H|^2\ \text{(magnetic loss power density)}$$

$$P_i = \frac{1}{2}(\omega\mu')|H|^2\ \text{(stored magnetic power density)}$$

# 3. Poynting's Theorem

- Power balance equation in electromagnetic field

$$P_s = P_r + P_l + j2\omega(W_m - W_e)$$

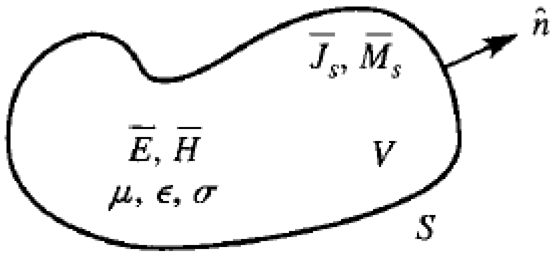
$$P_s = -\frac{1}{2}\int_V (\mathbf{E}\cdot\mathbf{J}_s^* + \mathbf{H}^*\cdot\mathbf{M}_s^*)dV : \text{ source power in } V$$

$$P_r = \frac{1}{2}\mathrm{Re}\oint_S \mathbf{E}\times\mathbf{H}^*\cdot d\mathbf{S} : \text{ power radiated through } S$$

$$P_l = \frac{1}{2}\int_V [(\sigma + \omega\varepsilon'')E^2 + \omega\mu'' H^2]dV : \text{ power loss in } V$$

$$W_m = \frac{1}{4}\int_V \mu' H^2 dV : \text{ stored magnetic energy}$$

$$W_e = \frac{1}{4}\int_V \varepsilon' E^2 dV : \text{ stored electric energy}$$



# 4. Impedance and Admittance

- Impedance and admittance in terms of terminal current and terminal voltage

$$Z_{\text{in}} = \frac{2P_{\text{in}}}{|I|^2} = \frac{P_r + P_l + j2\omega(W_m - W_e)}{|I|^2/2}: \text{ input impedance}$$

$I$ : terminal current

$$Y_{\text{in}} = \frac{2P_{\text{in}}}{|V|^2} = \frac{P_r + P_l + j2\omega(W_m - W_e)}{|V|^2/2}: \text{ input admittance}$$

$V$ : terminal current

## 5. Wave Equation

- Electromagnetic field in charge-free region

$\rho = 0$ (in charge-free region)

$$\nabla \cdot \mathbf{E} = 0, \ \nabla \times \mathbf{E} = -j\omega\mu\mathbf{H}$$

$$\nabla \cdot \mathbf{H} = 0, \ \nabla \times \mathbf{H} = j\omega\varepsilon\mathbf{E}$$

$$\nabla \times \nabla \times \mathbf{E} = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2\mathbf{E} = -j\omega\mu\nabla \times \mathbf{H} = \omega^2 \mu\varepsilon\mathbf{E}$$

$(\nabla^2 + \omega^2\mu\varepsilon)\mathbf{E} = 0$: wave equation for electric field

Similarly,

$(\nabla^2 + \omega^2\mu\varepsilon)\mathbf{H} = 0$: wave equation for magnetic field

# 6. Planewave

- Planewave: one-dimensional wave

$$\frac{\partial}{\partial x} = 0, \frac{\partial}{\partial y} = 0, \frac{\partial}{\partial z} \neq 0$$

$$\nabla \cdot \mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0 \rightarrow E_z = 0$$

$$\nabla^2 = \frac{d^2}{dz^2}$$

$$(\nabla^2 + \omega^2 \mu\varepsilon)\mathbf{E} = \left( \frac{d^2}{dz^2} + \omega^2 \mu\varepsilon \right)\mathbf{E} = 0$$

$$\mathbf{E} = \mathbf{E}^+ e^{-jkz} + \mathbf{E}^- e^{+jkz}$$

$\mathbf{E}^+ e^{-jkz}$ : wave propagating in $+z$ direction

$\mathbf{E}^- e^{+jkz}$ : wave propagating in $-z$ direction

$k = \omega\sqrt{\mu\varepsilon} = \dfrac{2\pi}{\lambda}$ $(\mu, \varepsilon$ real) : propagation constant

$\lambda$: wavelength

- Intrinsic impedance

$$\mathbf{E} = E_x \hat{\mathbf{x}} e^{-jkz}$$

$\hat{\mathbf{k}} = \hat{\mathbf{z}}$ : direction of propagation

$$\mathbf{H} = -\frac{1}{j\omega\mu}\nabla \times \mathbf{E} = -\frac{1}{j\omega\mu}\left(\hat{\mathbf{z}}\frac{\partial}{\partial z}\right) \times \mathbf{E}$$

$$= -\frac{1}{j\omega\mu}\hat{\mathbf{y}}(-jk)E_x e^{-jkz}$$

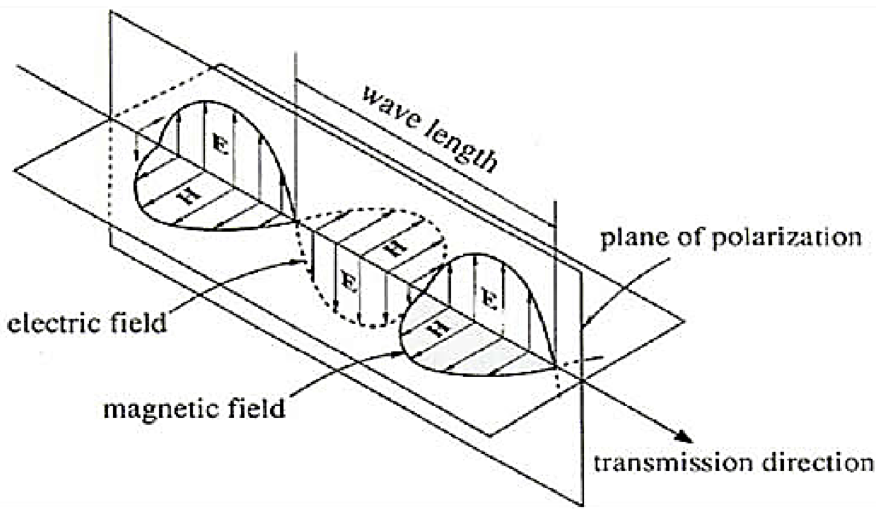
$$\mathbf{H} = \frac{k}{\omega\mu}E_x \hat{\mathbf{y}} e^{-jkz} = \frac{1}{\eta}E_x \hat{\mathbf{y}} e^{-jkz}$$

$$\mathbf{H} = \frac{\hat{\mathbf{k}} \times \mathbf{E}}{\eta}$$

$$\eta = \frac{\omega\mu}{k} = \sqrt{\frac{\mu}{\varepsilon}}$$ : intrinsic impedance

$$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 376.73 \ \Omega$$

$$\mathbf{E} = \eta \mathbf{H} \times \hat{\mathbf{k}}$$



# 7. Coding Example

Planewave

Input: $f$, $u_r$, $\varepsilon_r$

Calculate: $k$, $\lambda$, $\eta$

$$\omega = 2\pi f$$

$$\mu = \mu_0 \mu_r,\ \mu_0 = 4\pi \times 10^{-7}\ \text{H/m}$$

$$\varepsilon = \varepsilon_0 \varepsilon_r,\ \varepsilon_0 = 8.854 \times 10^{-12}\ \text{F/m}$$

$$k = \omega \sqrt{\mu \varepsilon}$$

$$\lambda = 2\pi / k$$

$$\eta = \sqrt{\frac{\mu}{\varepsilon}}$$

Fin

(End)