

Lecture 5: Polarization and Related Antenna Parameters

(Polarization of EM fields – revision. Polarization vector. Antenna polarization. Polarization loss factor and polarization efficiency.)

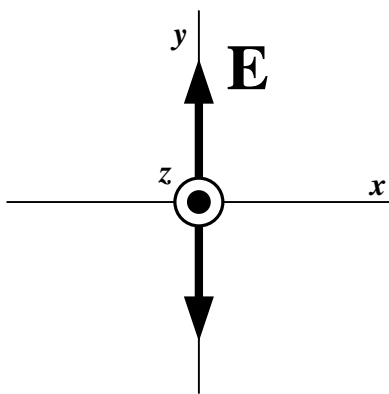
1. Introduction and Definition

The polarization of the EM field describes the orientation of its vectors at a given point and how it varies with time. In other words, it describes the way the direction and magnitude of the field vectors (usually \mathbf{E}) change in time. Polarization is associated with TEM time-harmonic waves where the \mathbf{H} vector relates to the \mathbf{E} vector simply by $\mathbf{H} = \hat{\mathbf{r}} \times \mathbf{E} / \eta$.

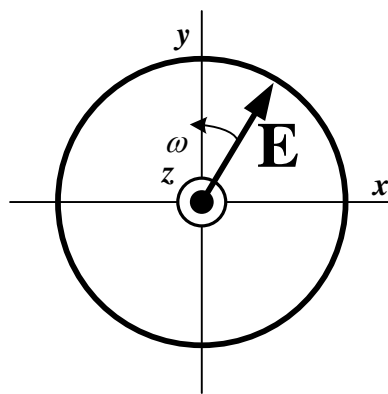
In antenna theory, we are concerned with the polarization of the field in the plane orthogonal to the direction of propagation—this is the plane defined by the far-zone vectors \mathbf{E} and \mathbf{H} . Remember that the far field is a quasi-TEM field.

The *polarization* is the locus traced by the extremity of the time-varying field vector at a fixed observation point.

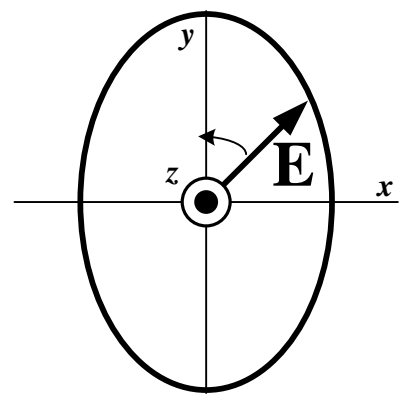
According to the shape of the trace, three types of polarization exist for harmonic fields: *linear*, *circular* and *elliptical*. Any polarization can be represented by two orthogonal linear polarizations, (E_x, E_y) or (E_H, E_V) , the fields of which may, in general, have different magnitudes and may be out of phase by an angle δ_L .



(a) linear polarization

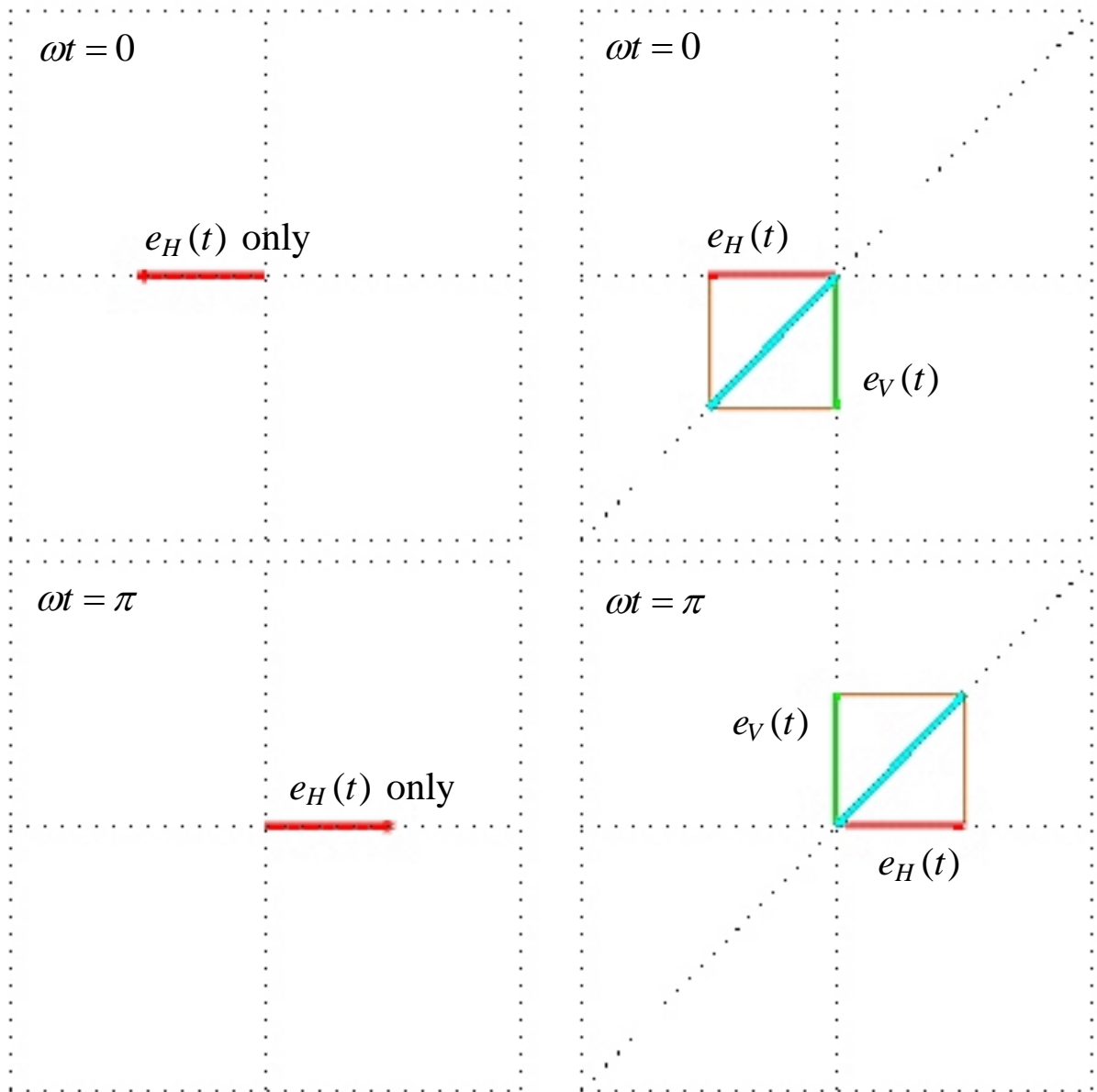


(b) circular polarization



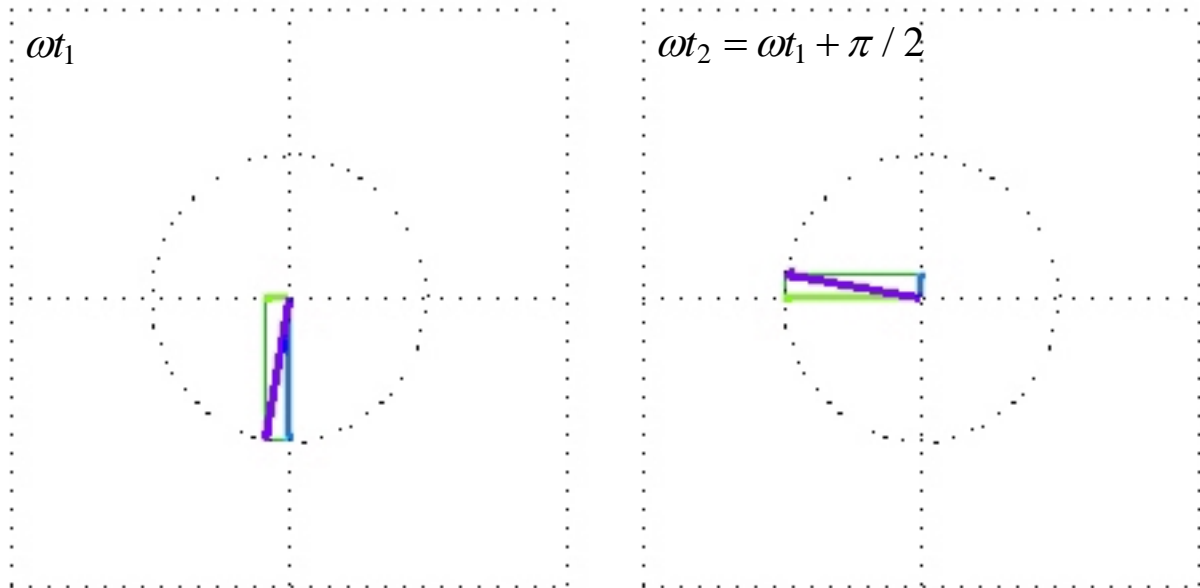
(c) elliptical polarization

- If $\delta_L = 0$ or $n\pi$, then the field is linearly polarized.



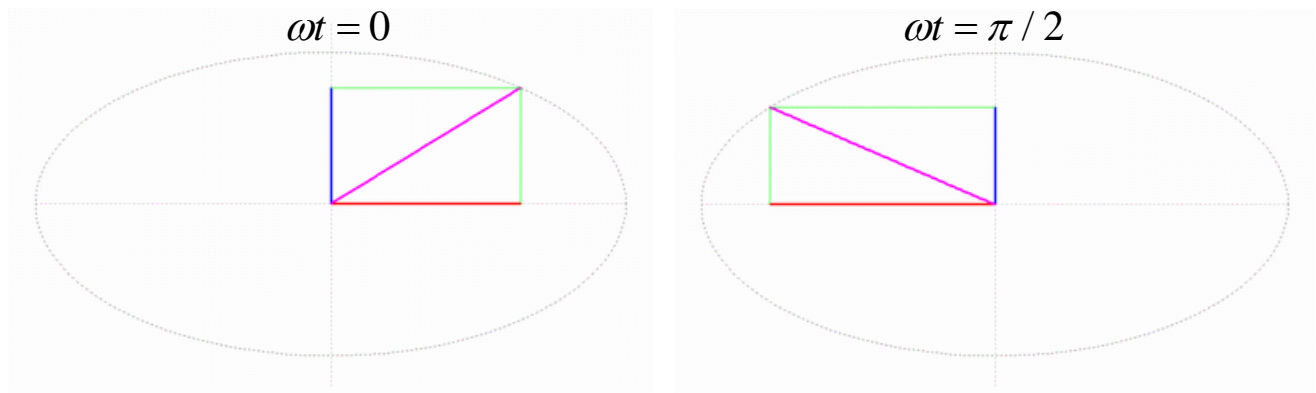
Animation: Linear Polarization, $\delta_L = 0$, $E_x = E_y$

- If $\delta_L = \pi / 2$ (90°) and $|E_x| = |E_y|$, then the field is circularly polarized.



Animation: Clockwise Circular Rotation

- In the most general case, the polarization is elliptical.



Animation: Counter-clockwise Elliptical Rotation

It is also true that any type of polarization can be represented by a right-hand circular and a left-hand circular polarizations (E_L , E_R). **[Animation]**

Next, we review the above statements and definitions, and introduce the new concept of polarization vector.

2. Field Polarization in Terms of Two Orthogonal Linearly Polarized Components

The polarization of any field can be represented by a set of two orthogonal linearly polarized fields. Assume that locally a far-field wave propagates along the z -axis. The far-zone field vectors have only transverse components. Then, the set of two orthogonal linearly polarized fields along the x -axis and along the y -axis, is sufficient to represent any TEM_z field. We use this arrangement to introduce the concept of *polarization vector*.

The field (time-dependent or phasor vector) is decomposed into two orthogonal components:

$$\mathbf{e} = \mathbf{e}_x + \mathbf{e}_y \Rightarrow \mathbf{E} = \mathbf{E}_x + \mathbf{E}_y, \quad (5.1)$$

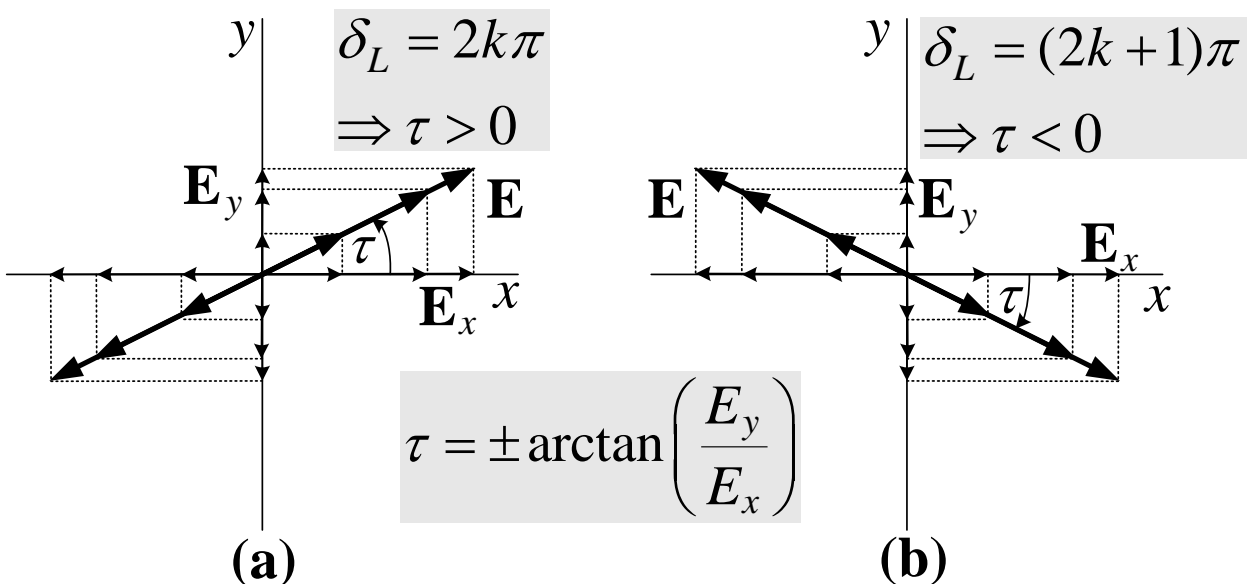
$$\begin{aligned} \mathbf{e}_x &= E_x \cos(\omega t - \beta z) \hat{\mathbf{x}} & \Rightarrow & \mathbf{E}_x = E_x e^{-j\beta z} \hat{\mathbf{x}} \\ \mathbf{e}_y &= E_y \cos(\omega t - \beta z + \delta_L) \hat{\mathbf{y}} & \Rightarrow & \mathbf{E}_y = E_y e^{-j\beta z} e^{j\delta_L} \hat{\mathbf{y}}. \end{aligned} \quad (5.2)$$

At a fixed position (assume $z = 0$), equation (5.1) can be written as

$$\begin{aligned} \mathbf{e}(t) &= \hat{\mathbf{x}} \cdot E_x \cos \omega t + \hat{\mathbf{y}} \cdot E_y \cos(\omega t + \delta_L) \\ &\Rightarrow \boxed{\mathbf{E} = \hat{\mathbf{x}} \cdot E_x + \hat{\mathbf{y}} \cdot E_y e^{j\delta_L}} \end{aligned} \quad (5.3)$$

Case 1: Linear polarization: $\delta_L = n\pi$, $n = 0, 1, 2, \dots$

$$\begin{aligned} \mathbf{e}(t) &= \hat{\mathbf{x}} \cdot E_x \cos(\omega t) + \hat{\mathbf{y}} \cdot E_y \cos(\omega t \pm n\pi) \\ &\Rightarrow \boxed{\mathbf{E} = \hat{\mathbf{x}} \cdot E_x \pm \hat{\mathbf{y}} \cdot E_y} \end{aligned} \quad (5.4)$$



Case 2: Circular polarization:

$$E_x = E_y = E_m \quad \text{and} \quad \delta_L = \pm \left(\frac{\pi}{2} + n\pi \right), \quad n = 0, 1, 2, \dots$$

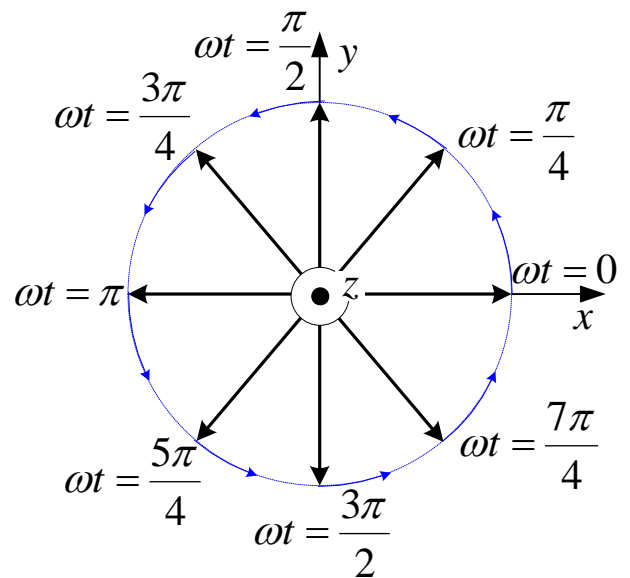
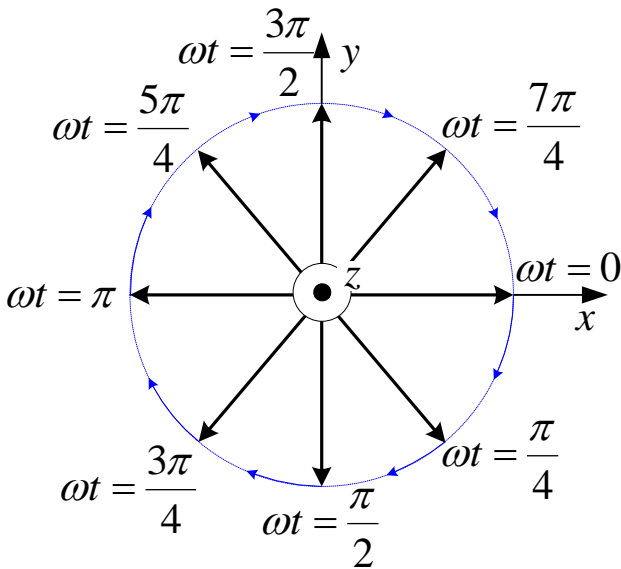
$$\mathbf{e}(t) = \hat{\mathbf{x}}E_x \cos(\omega t) + \hat{\mathbf{y}}E_y \cos[\omega t \pm (\pi / 2 + n\pi)]$$

$$\Rightarrow \boxed{\mathbf{E} = E_m(\hat{\mathbf{x}} \pm j\hat{\mathbf{y}})}$$

(5.5)

$$\boxed{\begin{aligned} \mathbf{E} &= E_m(\hat{\mathbf{x}} + j\hat{\mathbf{y}}) \\ \delta_L &= +\frac{\pi}{2} + 2n\pi \end{aligned}}$$

$$\boxed{\begin{aligned} \mathbf{E} &= E_m(\hat{\mathbf{x}} - j\hat{\mathbf{y}}) \\ \delta_L &= -\frac{\pi}{2} - 2n\pi \end{aligned}}$$

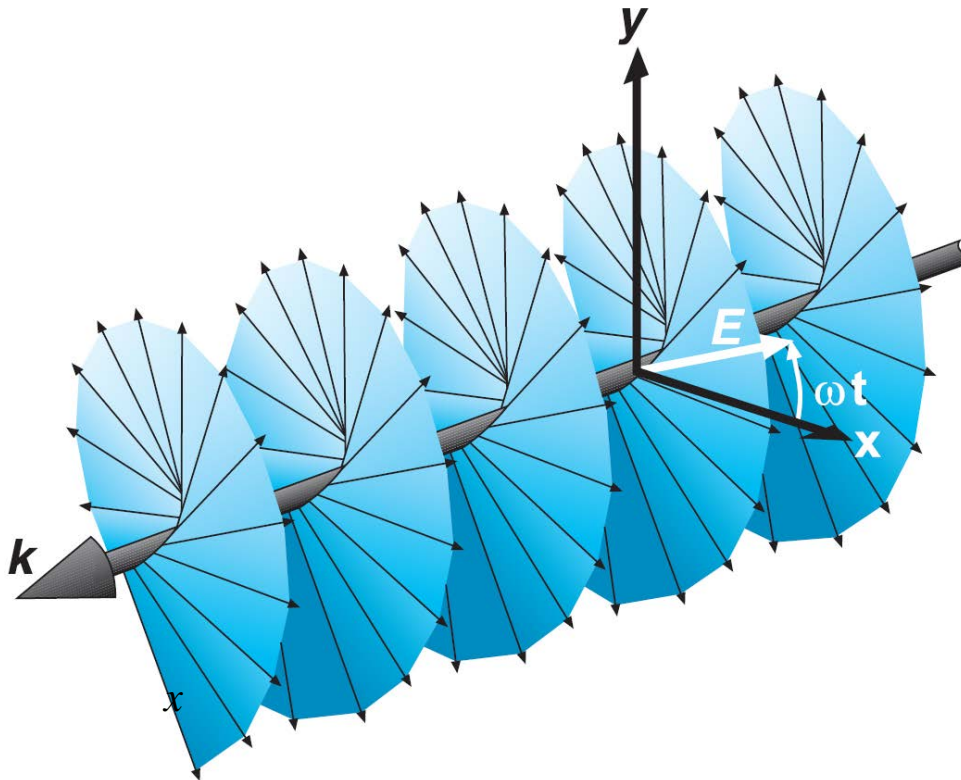


If $+\hat{\mathbf{z}}$ is the direction of propagation: **counterclockwise (CCW)** or **left-hand (LH)** polarization

If $+\hat{\mathbf{z}}$ is the direction of propagation: **clockwise (CW)** or **right-hand (RH)** polarization

Note that the sense of rotation changes if the direction of propagation changes. In the example above, if the wave propagates along $-\hat{\mathbf{z}}$, the plot to the left, where $\mathbf{E} = E_m(\hat{\mathbf{x}} + j\hat{\mathbf{y}})$, corresponds to a right-hand (RH) wave, while the plot to the right, where $\mathbf{E} = E_m(\hat{\mathbf{x}} - j\hat{\mathbf{y}})$, corresponds to a left-hand (LH) wave. Vice versa, if the wave propagates along $+\hat{\mathbf{z}}$, then the left plot shows a LH wave, whereas the right plot shows a RH wave.

A snapshot of the field vector along the axis of propagation is given below for a right-hand circularly polarized (RHCP) wave. Pick an observing position along the axis of propagation (see the plane defined by the x and y axes in the plot below) and imagine that the whole helical trajectory of the tip of the field vector moves along the wave vector \mathbf{k} . Are you going to see the vector rotating clockwise or counter-clockwise as you look along \mathbf{k} ? (Ans.: Clockwise, which is equivalent to RH sense of rotation.)



[Hayt, Buck, Engineering Electromagnetics, 8th ed., p. 399]

Case 3: Elliptic polarization

The field vector at a given point traces an ellipse as a function of time. This is the most general type of polarization, obtained for any phase difference δ and any ratio (E_x / E_y) . Mathematically, the linear and the circular polarizations are special cases of the elliptical polarization. In practice, however, the term *elliptical polarization* is used to indicate polarizations *other than linear or circular*.

$$\begin{aligned} \mathbf{e}(t) &= \hat{\mathbf{x}}E_x \cos \omega t + \hat{\mathbf{y}}E_y \cos(\omega t + \delta_L) \\ \Rightarrow \mathbf{E} &= \hat{\mathbf{x}}E_x + \hat{\mathbf{y}}E_y e^{j\delta_L} \end{aligned} \quad (5.6)$$

Show that the trace of the time-dependent vector is an ellipse:

$$\begin{aligned}
 e_y(t) &= E_y(\cos \omega t \cdot \cos \delta_L - \sin \omega t \cdot \sin \delta_L) \\
 \cos \omega t &= \frac{e_x(t)}{E_x} \text{ and } \sin \omega t = \sqrt{1 - \left(\frac{e_x(t)}{E_x}\right)^2} \\
 \sin^2 \delta_L &= \left[\frac{e_x(t)}{E_x}\right]^2 - 2\left[\frac{e_x(t)}{E_x}\right]\left[\frac{e_y(t)}{E_y}\right]\cos \delta_L + \left[\frac{e_y(t)}{E_y}\right]^2
 \end{aligned}$$

or (dividing both sides by $\sin^2 \delta_L$),

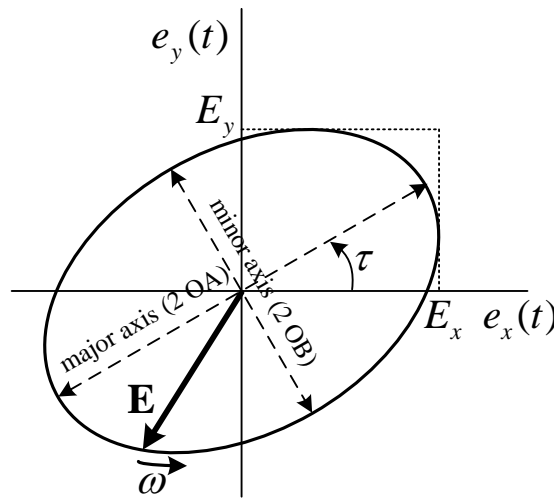
$$1 = x^2(t) - 2x(t)y(t)\cos \delta_L + y^2(t), \quad (5.7)$$

where

$$\begin{aligned}
 x(t) &= \frac{e_x(t)}{E_x \sin \delta_L} = \frac{\cos \omega t}{\sin \delta_L}, \\
 y(t) &= \frac{e_y(t)}{E_y \sin \delta_L} = \frac{\cos(\omega t + \delta_L)}{\sin \delta_L}.
 \end{aligned}$$

Equation (5.7) is the equation of an ellipse centered in the xy plane. It describes the trajectory of a point of coordinates $x(t)$ and $y(t)$, i.e., normalized $e_x(t)$ and $e_y(t)$ values, along an ellipse where the point moves with an angular frequency ω .

As the circular polarization, the elliptical polarization can be *right-handed* or *left-handed*, depending on the relation between the direction of propagation and the sense of rotation.



The parameters of the polarization ellipse are given below. Their derivation is given in Appendix I.

a) major axis ($2 \times \text{OA}$)

$$\text{OA} = \sqrt{\frac{1}{2} \left[E_x^2 + E_y^2 + \sqrt{E_x^4 + E_y^4 + 2E_x^2 E_y^2 \cos(2\delta_L)} \right]} \quad (5.8)$$

b) minor axis ($2 \times \text{OB}$)

$$\text{OB} = \sqrt{\frac{1}{2} \left[E_x^2 + E_y^2 - \sqrt{E_x^4 + E_y^4 + 2E_x^2 E_y^2 \cos(2\delta_L)} \right]} \quad (5.9)$$

c) tilt angle τ

$$\tau = \frac{1}{2} \arctan \left(\underbrace{\frac{2E_x E_y \cos \delta_L}{E_x^2 - E_y^2}}_A \right) \pm \frac{\pi}{2} \quad (5.10)$$

Note: Eq. (5.10) produces an infinite number of angles, $\tau = (\arctan A)/2 \pm n\pi/2$, $n = 1, 2, \dots$. Thus, it gives not only the angle which the major axis of the ellipse forms with the x axis but also the angle of the minor axis with the x axis. In spherical coordinates, τ is usually specified with respect to the $\hat{\theta}$ direction

d) axial ratio

$$\text{AR} = \frac{\text{major axis}}{\text{minor axis}} = \frac{\text{OA}}{\text{OB}} \quad (5.11)$$

Note: The linear and circular polarizations as special cases of the elliptical polarization:

- If $\delta_L = \pm \left(\frac{\pi}{2} + 2n\pi \right)$ and $E_x = E_y$, then $\text{OA} = \text{OB} = E_x = E_y$; the ellipse becomes a circle.
- If $\delta_L = n\pi$, then $\text{OB} = 0$ and $\tau = \pm \arctan(E_y / E_x)$; the ellipse collapses into a line.

3. Field Polarization in Terms of Two Circularly Polarized Components

The representation of a complex vector field in terms of circularly polarized components is somewhat less intuitive but it is actually more useful in the calculation of the polarization ellipse parameters. This time, the total field phasor is represented as the superposition of two circularly polarized waves,

one right-handed and the other left-handed. For the case of a wave propagating along $-z$ [see Case 2 and Eq. (5.5)],

$$\mathbf{E} = E_R(\hat{\mathbf{x}} + j\hat{\mathbf{y}}) + E_L(\hat{\mathbf{x}} - j\hat{\mathbf{y}}). \quad (5.12)$$

Here, E_R and E_L are, in general, complex phasors. Assuming a relative phase difference of $\delta_C = \varphi_R - \varphi_L$, one can write (5.12) as

$$\mathbf{E} = e_R e^{j\delta_C} (\hat{\mathbf{x}} + j\hat{\mathbf{y}}) + e_L (\hat{\mathbf{x}} - j\hat{\mathbf{y}}), \quad (5.13)$$

where e_R and e_L are real numbers.

The relations between the linear-component and the circular-component representations of the field polarization are easily found as

$$\mathbf{E} = \hat{\mathbf{x}} \underbrace{(E_R + E_L)}_{E_x} + \hat{\mathbf{y}} \underbrace{j(E_R - E_L)}_{E_y} \quad (5.14)$$

$$\Rightarrow \begin{cases} E_x = E_R + E_L \\ E_y = j(E_R - E_L) \end{cases} \quad (5.15)$$

$$\Rightarrow \begin{cases} E_R = 0.5(E_x - jE_y) \\ E_L = 0.5(E_x + jE_y). \end{cases} \quad (5.16)$$

4. Polarization Vector and Polarization Ratio of a Plane Wave

The *polarization vector* is the normalized phasor of the electric field vector. It is a complex-valued vector of unit magnitude, i.e., $\hat{\boldsymbol{\rho}}_L \cdot \hat{\boldsymbol{\rho}}_L^* = 1$.

$$\hat{\boldsymbol{\rho}}_L = \frac{\mathbf{E}}{E_m} = \hat{\mathbf{x}} \frac{E_x}{E_m} + \hat{\mathbf{y}} \frac{E_y}{E_m} e^{j\delta_L}, \quad E_m = \sqrt{E_x^2 + E_y^2} \quad (5.17)$$

The expression in (5.17) assumes a wave decomposition into linearly polarized (x and y) components, thereby the subscript L . Polarization vector in terms of RHCP and LHCP components is also used. The polarization vector defined in (5.17) takes the following specific forms in the cases of linearly, circularly and elliptically polarized waves.

Case 1: Linear polarization (the polarization vector is real-valued)

$$\hat{\boldsymbol{\rho}} = \hat{\mathbf{x}} \frac{E_x}{E_m} \pm \hat{\mathbf{y}} \frac{E_y}{E_m}, \quad E_m = \sqrt{E_x^2 + E_y^2} \quad (5.18)$$

where E_x and E_y are real numbers.

Case 2: Circular polarization (the polarization vector is complex-valued)

$$\hat{\mathbf{p}}_L = \frac{1}{\sqrt{2}}(\hat{\mathbf{x}} \pm j\hat{\mathbf{y}}), \quad E_m = \sqrt{2}E_x = \sqrt{2}E_y \quad (5.19)$$

The *polarization ratio* is the ratio of the phasors of the two orthogonal polarization components. In general, it is a complex number:

$$\tilde{r}_L = r_L e^{j\delta_L} = \frac{\tilde{E}_y}{\tilde{E}_x} = \frac{E_y e^{j\delta_L}}{E_x} \quad \text{or} \quad \tilde{r}_L = \frac{\tilde{E}_V}{\tilde{E}_H} \quad (5.20)$$

Point of interest: In the case of circular-component representation, the polarization ratio is defined as

$$\tilde{r}_C = r_C e^{j\delta_C} = \frac{\tilde{E}_R}{\tilde{E}_L} \quad (5.21)$$

The circular polarization ratio \tilde{r}_C is of particular interest since the axial ratio of the polarization ellipse AR can be expressed as

$$AR = \left| \frac{r_C + 1}{r_C - 1} \right| \quad (5.22)$$

Besides, its tilt angle with respect to the y (vertical) axis is simply

$$\tau_V = \delta_C / 2 + n\pi, \quad n = 0, \pm 1, \dots \quad (5.23)$$

Comparing (5.10) and (5.23) readily shows the relation between the phase difference δ_C of the circular-component representation and the linear polarization ratio $\tilde{r}_L = r_L e^{j\delta_L}$:

$$\delta_C = \arctan\left(\frac{2r_L}{1-r_L^2} \cos \delta_L\right) \quad (5.24)$$

We can calculate r_C from the linear polarization ratio \tilde{r}_L making use of (5.11) and (5.22):

$$AR = \frac{r_C + 1}{r_C - 1} = \sqrt{\frac{1 + r_L^2 + \sqrt{1 + r_L^4 + 2r_L^2 \cos(2\delta_L)}}{1 + r_L^2 - \sqrt{1 + r_L^4 + 2r_L^2 \cos(2\delta_L)}}} \quad (5.25)$$

Using (5.24) and (5.25) allows for the switching between the representation of the wave polarization in terms of linear and circular components.

5. Antenna Polarization

The *polarization of a transmitting antenna* is the polarization of its radiated wave in the far zone. *The polarization of a receiving antenna* is the polarization of an incident plane wave, which, for a given power flux density, results in maximum available power at the antenna terminals.

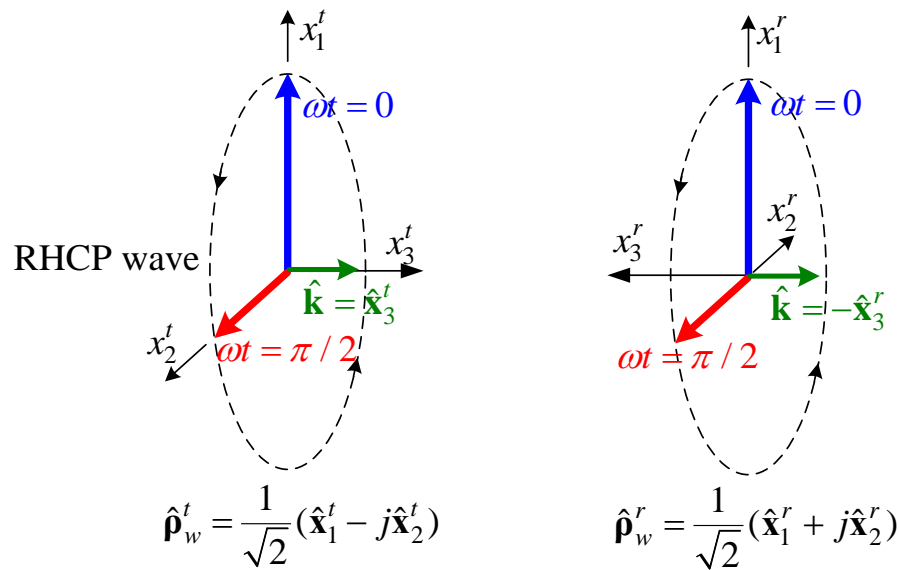
By convention, the antenna polarization is defined by the polarization vector of the wave it transmits. Therefore, the antenna polarization vector is determined according to the definition of antenna polarization in a transmitting mode. Notice that the polarization vector $\hat{\rho}_w^t$ of a wave in the coordinate system of transmission (the wave moves away from the antenna at the origin, i.e., along $\hat{\mathbf{r}}$) is the conjugate of its polarization vector $\hat{\rho}_w^r$ in the coordinate system of reception (the wave moves toward the antenna at the origin, i.e., along $-\hat{\mathbf{r}}$):

$$\hat{\rho}_w^r = (\hat{\rho}_w^t)^*. \quad (5.26)$$

The conjugation is without importance for a linearly polarized wave since its polarization vector is real. It is, however, important in the cases of circularly and elliptically polarized waves.

This is illustrated in the figure below with a RHCP wave. Let the coordinate triplet (x_1^t, x_2^t, x_3^t) represent the coordinate system of the transmitting antenna while (x_1^r, x_2^r, x_3^r) represents that of the receiving antenna. In antenna analysis, the plane of polarization is usually given in spherical coordinates by $(\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2) \equiv (\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}})$ and the third axis obeys $\hat{\mathbf{x}}_1 \times \hat{\mathbf{x}}_2 = \hat{\mathbf{x}}_3$, i.e., $\hat{\mathbf{x}}_3 = \hat{\mathbf{r}}$. Since the transmitting and receiving antennas face each other, their coordinate systems are oriented so that $\hat{\mathbf{x}}_3^t = -\hat{\mathbf{x}}_3^r$ (i.e., $\hat{\mathbf{r}}^r = -\hat{\mathbf{r}}^t$). If we align the axes $\hat{\mathbf{x}}_1^t$ and $\hat{\mathbf{x}}_1^r$, then $\hat{\mathbf{x}}_2^t = -\hat{\mathbf{x}}_2^r$ must hold. This changes the sign of the respective (2nd) field vector component. Upon normalization, this results in a change of sign in the imaginary part of the wave polarization vector.

Bearing in mind the definitions of antenna polarization in transmitting and receiving modes, we conclude that *in a common coordinate system the transmitting-mode polarization vector of an antenna is the conjugate of its receiving-mode polarization vector.*



6. Polarization Loss Factor (Polarization Efficiency)

Generally, the polarization of the receiving antenna is not the same as the polarization of the incident wave. This is called *polarization mismatch*. The polarization loss factor (PLF) characterizes the loss of EM power due to the polarization mismatch. The PLF is defined so that it attains a value of 1 (or 100%, or 0 dB) if there is no polarization mismatch, i.e., the antenna receives the maximum possible power for the given incident power density. A PLF equal to 0 ($-\infty$ dB) indicates complete polarization mismatch and inability to capture power from the incident wave. Thus,

$$0 \leq \text{PLF} \leq 1. \quad (5.27)$$

Note that the polarization loss has nothing to do with dissipation. It can be viewed as a “missed opportunity” to capture as much power from the incident wave as possible. The *polarization efficiency* has the same meaning as PLF.

Let us denote the polarization vector of a wave incident upon a receiving antenna as $\hat{\mathbf{p}}_w$. In the coordinate system where the receiving antenna is at the origin, this vector describes a wave propagating along $-\hat{\mathbf{r}}$. Assume also that the polarization vector of the wave that the receiving antenna would produce if it were to operate in transmitting mode is $\hat{\mathbf{p}}_a$. In the coordinate system where the receiving antenna is at the origin, this vector describes a wave propagating along $+\hat{\mathbf{r}}$. Then, the PLF is defined as

$$\text{PLF} = |\hat{\mathbf{p}}_w \cdot \hat{\mathbf{p}}_a|^2. \quad (5.28)$$

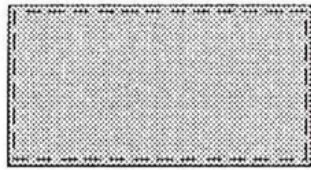
Note that if $\hat{\mathbf{p}}_w^* = \hat{\mathbf{p}}_a$, i.e., the incident field is

$$\mathbf{E}^i = E_m \hat{\mathbf{p}}_a^*$$

PLF = 1, and we obtain maximum possible received power at the antenna terminals. Remember that the transmitting-mode and receiving-mode polarization vectors of a wave are mutually conjugate? This means that $\hat{\mathbf{p}}_a^*$ is nothing but the wave the receiving antenna would generate if it were to transmit in the direction of the incident-wave propagation. Thus, the optimal polarization of the incident wave is the one that matches the polarization of the wave produced by the receiving antenna if it was the one launching the incident wave.

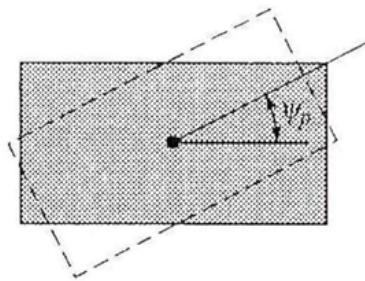
Here are some simple examples:

- 1) if $\hat{\mathbf{p}}_w = \hat{\mathbf{p}}_a^* = \hat{\mathbf{p}}_a = \hat{\mathbf{x}}$, then PLF=1;
- 2) if $\hat{\mathbf{p}}_w = \hat{\mathbf{x}}$ and $\hat{\mathbf{p}}_a = \hat{\mathbf{p}}_a^* = \hat{\mathbf{y}}$, then PLF=0;
- 3) if $\hat{\mathbf{p}}_w = \hat{\mathbf{p}}_a = \hat{\mathbf{x}} + j\hat{\mathbf{y}}$, then PLF=0;
- 4) if $\hat{\mathbf{p}}_w = \hat{\mathbf{x}} + j\hat{\mathbf{y}}$ and $\hat{\mathbf{p}}_a = \hat{\mathbf{x}} - j\hat{\mathbf{y}}$ ($\hat{\mathbf{p}}_w^* = \hat{\mathbf{p}}_a$), then PLF=1.



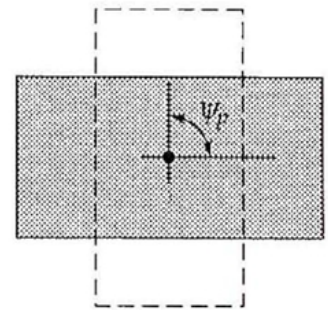
$$\text{PLF} = |\hat{\mathbf{p}}_w \cdot \hat{\mathbf{p}}_a|^2 = 1$$

(aligned)



$$\text{PLF} = |\hat{\mathbf{p}}_w \cdot \hat{\mathbf{p}}_a|^2 = \cos^2 \psi_p$$

(rotated)



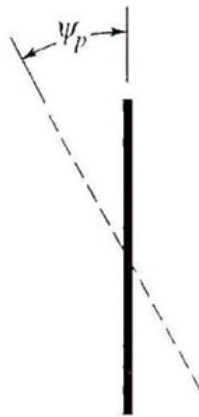
$$\text{PLF} = |\hat{\mathbf{p}}_w \cdot \hat{\mathbf{p}}_a|^2 = 0$$

(orthogonal)



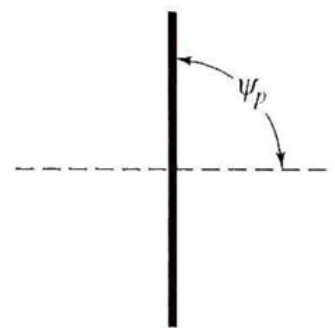
$$\text{PLF} = |\hat{\mathbf{p}}_w \cdot \hat{\mathbf{p}}_a|^2 = 1$$

(aligned)



$$\text{PLF} = |\hat{\mathbf{p}}_w \cdot \hat{\mathbf{p}}_a|^2 = \cos^2 \psi_p$$

(rotated)



$$\text{PLF} = |\hat{\mathbf{p}}_w \cdot \hat{\mathbf{p}}_a|^2 = 0$$

(orthogonal)

[Balanis, 2nd ed.]

In a communication link, the PLF has to be expressed by the polarization vectors of the transmitting and receiving antennas, $\hat{\rho}_{Tx}$ and $\hat{\rho}_{Rx}$, respectively. Both of these are defined in the coordinate system of the respective antenna as the polarization of the transmitted wave. However, these two coordinate systems have their radial unit vectors pointing in opposite directions, i.e., $\hat{\mathbf{r}}_{Rx} = -\hat{\mathbf{r}}_{Tx}$ as illustrated in the figure below. Therefore, either $\hat{\rho}_{Tx}$ or $\hat{\rho}_{Rx}$ has to be conjugated when calculating the PLF (it does not matter which one). For example, if the reference coordinate system is that of the receiving antenna, then

$$\text{PLF} = \left| \hat{\rho}_{Tx}^* \cdot \hat{\rho}_{Rx} \right|^2. \quad (5.29)$$

The expression $\text{PLF} = \left| \hat{\rho}_{Tx} \cdot \hat{\rho}_{Rx}^* \right|^2$ is also correct.



Examples

Example 5.1. The electric field of a linearly polarized EM wave is

$$\mathbf{E}^i = \hat{\mathbf{x}} \cdot E_m(x, y) e^{-j\beta z}.$$

It is incident upon a linearly polarized receiving antenna, which would transmit the field

$$\mathbf{E}_a = (\hat{\mathbf{x}} + \hat{\mathbf{y}}) \cdot e^{j\beta z}$$

if it were to operate in a transmitting instead of receiving mode. Find the PLF.

Notice that \mathbf{E}_a propagates along $-z$ in accordance with the requirement that it represents a transmitted wave.

$$\text{PLF} = \left| \hat{\mathbf{x}} \cdot \frac{1}{\sqrt{2}} (\hat{\mathbf{x}} + \hat{\mathbf{y}}) \right|^2 = \frac{1}{2}$$

$$\text{PLF}_{[\text{dB}]} = 10 \log_{10} 0.5 = -3 \text{ dB}$$

Example 5.2. A transmitting antenna produces a far-zone field, which is RH circularly polarized. This field impinges upon a receiving antenna, whose polarization (in transmitting mode) is also RH circular. Determine the PLF.

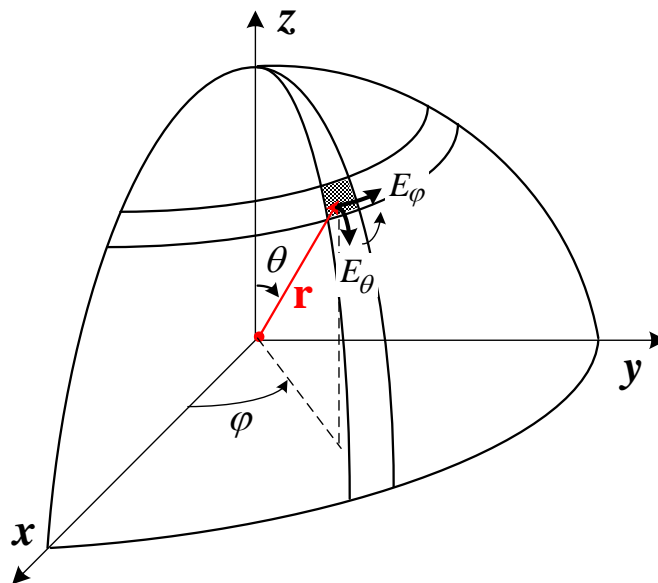
Both antennas (the transmitting one and the receiving one) are RH circularly polarized in transmitting mode. Assume that a transmitting antenna is located at the center of a spherical coordinate system. The far-zone field it would produce is described as

$$\mathbf{E}^{far} = E_m \left[\hat{\boldsymbol{\theta}} \cdot \cos \omega t + \hat{\boldsymbol{\phi}} \cdot \cos(\omega t - \pi / 2) \right].$$

This is a RHCP field with respect to the outward radial direction $\hat{\mathbf{r}}$. Its polarization vector is

$$\hat{\boldsymbol{\rho}}_{Tx} = \frac{\hat{\boldsymbol{\theta}} - j\hat{\boldsymbol{\phi}}}{\sqrt{2}}.$$

This is exactly the polarization vector of the transmitting antenna in its own coordinate system.



Since the receiving antenna is also RHCP, its polarization vector is

$$\hat{\boldsymbol{\rho}}_{Rx} = \frac{\hat{\boldsymbol{\theta}} - j\hat{\boldsymbol{\phi}}}{\sqrt{2}}.$$

The PLF is calculated as per (5.29):

$$\text{PLF} = |\hat{\mathbf{p}}_{\text{Tx}}^* \cdot \hat{\mathbf{p}}_{\text{Rx}}|^2 = \frac{|(\hat{\boldsymbol{\theta}} + j\hat{\boldsymbol{\phi}}) \cdot (\hat{\boldsymbol{\theta}} - j\hat{\boldsymbol{\phi}})|^2}{4} = 1,$$

$$\text{PLF}_{[\text{dB}]} = 10 \log_{10} 1 = 0.$$

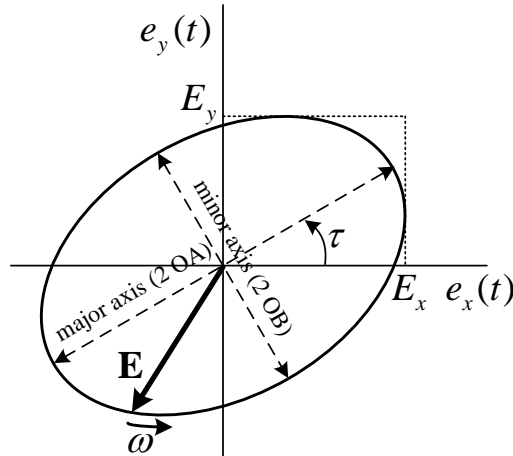
There is no polarization loss.

Exercise: Show that an antenna of RH circular polarization (in transmitting mode) cannot receive LH circularly polarized incident wave (or a wave emitted by a left-circularly polarized antenna).

Appendix I

Find the tilt angle τ , the length of the major axis OA, and the length of the minor axis OB of the ellipse described by the equation:

$$\sin^2 \delta = \left[\frac{e_x(t)}{E_x} \right]^2 - 2 \left[\frac{e_x(t)}{E_x} \right] \left[\frac{e_y(t)}{E_y} \right] \cos \delta + \left[\frac{e_y(t)}{E_y} \right]^2. \quad (\text{A-1})$$



Equation (A-1) can be written as

$$a \cdot x^2 - b \cdot xy + c \cdot y^2 = 1, \quad (\text{A-2})$$

where

$x = e_x(t)$ and $y = e_y(t)$ are the coordinates of a point of the ellipse centered in the xy plane;

$$a = \frac{1}{E_x^2 \sin^2 \delta};$$

$$b = \frac{2 \cos \delta}{E_x E_y \sin^2 \delta};$$

$$c = \frac{1}{E_y^2 \sin^2 \delta}.$$

After dividing both sides of (A-2) by (xy) , one obtains

$$a \frac{x}{y} - b + c \frac{y}{x} = \frac{1}{xy}. \quad (\text{A-3})$$

Introducing $\xi = \frac{y}{x} = \frac{e_y(t)}{e_x(t)}$, one obtains that

$$x^2 = \frac{1}{c\xi^2 - b\xi + a}$$

$$\Rightarrow \rho^2(\xi) = x^2 + y^2 = x^2(1 + \xi^2) = \frac{1 + \xi^2}{c\xi^2 - b\xi + a}. \quad (\text{A-4})$$

Here, ρ is the distance from the center of the coordinate system to the point on the ellipse. We want to know at what values of ξ the maximum and the minimum of ρ occur (ξ_{\min} , ξ_{\max}). This will produce the tilt angle τ . We also want to know the values of ρ_{\max} (major axis) and ρ_{\min} (minor axis). Then, we have to solve

$$\frac{d(\rho^2)}{d\xi} = 0, \text{ or}$$

$$\xi_m^2 - \frac{2(a-c)}{b} \xi_m - 1 = 0, \text{ where } \xi_m \equiv \xi_{\min}, \xi_{\max}. \quad (\text{A-5})$$

(A-5) is solved for the angle τ , which relates to ξ_{\max} as

$$\xi_{\max} = \tan \tau = (y/x)_{\max}. \quad (\text{A-6})$$

Substituting (A-6) in (A-5) yields:

$$\left(\frac{\sin \tau}{\cos \tau} \right)^2 - 2C \left(\frac{\sin \tau}{\cos \tau} \right) - 1 = 0 \quad (\text{A-7})$$

where

$$C = \frac{a - c}{b} = \frac{E_y^2 - E_x^2}{2E_x E_y \cos \delta}.$$

Multiplying both sides of (A-7) by $\cos^2 \tau$ and re-arranging results in

$$\underbrace{\cos^2 \tau - \sin^2 \tau}_{\cos(2\tau)} + \underbrace{2C \sin \tau \cdot \cos \tau}_{C \sin(2\tau)} = 0.$$

Thus, the solution of (A-7) is

$$\tan(2\tau) = -1 / C$$

or

$$\tau_1 = \frac{1}{2} \arctan \left(\frac{2E_x E_y \cos \delta}{E_x^2 - E_y^2} \right); \quad \tau_2 = \tau_1 + \frac{\pi}{2}. \quad (\text{A-8})$$

The angles τ_1 and τ_2 are the angles between the major and minor axes with the x axis. Substituting τ_1 and τ_2 back in ρ (see A-4) yields the expressions for OA and OB.