Test of Impedance Formulas for Small Circular Loop Antennas

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Abstract

In this paper, two simple formulas are tested for the impedance of small circular loop antennas against the theoretical values and the numerical simulation. The first formula is a textbook formula consisting of equations for the radiation resistance and the loop inductance. The second formula is an empirical formula proposed by Awadalla and Sharshar. The accuracy of the two formulas are examined using the full-wave simulation using the CST Studio Suite™ and theoretical results by Strorer. It has been found that the two formulas give rough approximate impedance values for loop antennas whose perimeter is less than a few tenths of a wavelength. The reasons for large discrepancy of the formulas with accurate values are suggested.

Keywords: Loop antenna, impedance, formulas, numerical simulation

I. INTRODUCTION

The circular loop is one of the fundamental antenna elements that merits thorough theoretical investigation [1]. In using a circular loop antenna, one is encountered with the task of impedance matching and a simple impedance formula would be of great help.

The impedance characteristics as well as the radiation pattern of a circular loop antenna are dictated by its current distribution. Antenna textbooks describe loop antennas of constant current. Theoretical models for the current distribution on a loop antenna current include uniform, sinusoidal, hyperbolic cosine, Fourier cosine and traveling-wave types [2]. Earlier investigation into the circular loop antenna has mostly been analytical involving cylindrical functions [3]-[7], from which a formula for the input impedance is derived. Analytical formulas for the input impedance of a loop are quite complicated and computer calculations are required for their evaluation. Accurate current distribution on a circular loop antenna can also be obtained using numerical method [8]-[9], from which empirical formulas might be derived.

Notwithstanding the accurate loop current, the finding of an accurate impedance formula is not simple since involves the integration of the Bessel functions [6], [10]. Simple formulas resort to a small loop approximation [1] or an empirical approach based on a transmission line model and the curve-fitting of numerical results [11].

In this paper, two simple formulas for the impedance of small circular loop antennas are tested for their accuracy and range of validity. The first formula tested is a textbook formula [1] consisting of the radiation resistance and the inductance of an electrically small circular loop. The second formula is the one proposed by Adawalla and Sharshar [11] in 1984 based on the transmission line theory and the curve-fitting of numerical results. For the test of the two formulas, the widely used electromagnetic simulation software CST Studio Suite™ and theoretical results by Strorer [4] have been employed.

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II. IMPEDANCE FORMULAS

Fig. 1 shows the geometry of a loop antenna where \( a \) is the wire radius, \( b \) is the mean radius of the loop, and \( g \) is the feed gap width. The wire is of perfectly conducting material so that its ohmic resistance is zero.

A textbook formula for the input resistance is obtained by assuming a small loop and a uniform current around the loop and by calculating the radiation resistance, which is same as the input resistance. It is given by

\[
R_i = 20(\beta^2 S)^2
\]  

(1)

where

\[
\beta = 2\pi/\lambda
\]

and

\[
S = \pi b^2
\]

In the textbook formula, the reactance of a small loop antenna is modeled using the inductance of a loop with uniform current which is given by [11]

\[
L = \mu \sqrt{b(b-a)} \left[ \frac{2}{k} - k \right] K(k) - \frac{2}{k} E(k)
\]

(2)

where

\[
k^2 = \frac{4b(b-a)}{(2b-a)^2}
\]

and \( K \) and \( E \) are the complete elliptic integrals of the first and second kinds. For thin-wire loops where \( b \gg a \)

\[
L = \mu b \left[ \ln \frac{8b}{a} - 2 \right]
\]

(3)

the inductance is simplified to

\[
Z = R + j\omega L
\]

(5)

The second formula investigated in this paper is proposed by Adawalla and Sharshar [11], who claim their formulas are valid up to \( kb = 0.5 \) for the input resistance and up to \( kb = 0.8 \) for the input reactance. The real part of their impedance formula is obtained by curve-fitting the numerical results while the imaginary part is obtained by modeling the loop as a short-circuited parallel-wire transmission line, where the area formed by two wires is same as that of the loop.

The real part of the Adawall-Sharshar formula is given by

\[
R = a\tan^2(kP/2)
\]

(6)

where

\[
P = 2\pi b
\]

\[
a = \begin{cases} 
1.793, & P/\lambda \leq 0.2 \\
1.722, & 0.2 \leq P/\lambda \leq 0.5 
\end{cases}
\]
\[
b = \begin{cases} 
3.928, & P/\lambda \leq 0.2 \\
3.676, & 0.2 \leq P/\lambda \leq 0.5 
\end{cases}
\]

while the imaginary part is given by

\[X = Z_0 \tan (\beta b)\]  \hspace{1cm} (7)

where

\[Z_0 = 276 \ln \frac{b}{a}\]

The impedance formula by Adawalla and Sharshar is now given by

\[Z = R + jX\]  \hspace{1cm} (8)

### III. TEST OF FORMULAS

Simple formulas (5) and (8) are for a small loop antenna with an infinitesimal feed gap. In the simulation using the CST Studio Suite™, a finite width feed gap is required to excite the loop antenna. With a finite-width feed gap, the effect is twofold. Firstly the total loop length is decreased by the gap width \(g\). Secondly the capacitance between gap surfaces adds a parallel loading to the input impedance of the loop.

To compensate for the gap effect in the numerical simulation, the radius \(b_c\) of the loop in the simulation model is adjusted so that the loop conductor length is same as the loop without a feed gap. This approximation is valid when the gap width is a small fraction of the loop length.

The adjusted loop radius is related to the radius \(b\) of a loop without a feed gap by the following equation.

\[2\pi b_c - g = 2\pi b\]

or

\[b_c = b + \frac{g}{2\pi}\]  \hspace{1cm} (9)

Figure 2 shows an equivalent circuit of a loop antenna with a finite width feed gap, where \(R\) and \(L\) is the resistance and inductance of a loop antenna without a feed gap and \(C\) is the gap capacitance.

The admittance \(Y_c\) of a loop with a feed gap is related to the admittance \(Y\) of a loop without a feed gap by the following equation

\[Y_c = Y + j\omega C\]

so that the admittance \(Y\) and impedance \(Z\) of a loop antenna without a feed gap are given by

\[Y = Y_c - j\omega C\]  \hspace{1cm} (10)

\[Z = \frac{1}{Y} = R + jX\]  \hspace{1cm} (11)

The effect of the feed gap is noticeable as the frequency approaches the first antiresonance where \(R\) and \(X\) go through a very rapid change over a narrow frequency span. \(Y_c\) is obtained by the simulation by the CST Studio Suite™ and used to obtain \(Y\) in (10).

Formulas (5) and (8) are tested using numerical simulation using the CST Studio Suite™ and theoretical results by Storer [4]. The CST Studio Suite™ provides three solvers applicable to the loop
antenna problem: Transient Solver (a time-domain solver), Frequency Domain Solver, and Integral Equation Solver. Among three solvers the Frequency Domain Solver gives most accurate results for the input impedance of the loop antenna.

The Storer's theoretical results are obtained from the series solution to the Hallén's integral equation for thin-wire loop antennas. The impedance formula involves a series summation whose coefficients are computed by integrating the Bessel functions. Storer provides detailed tables of the loop antenna impedance for \( kb/a \) values of 54.50, 90.02, 148.41, 244.69, and 403.43 and for \( kb \) values from 0.05 to 2.50. The case \( kb/a = 54.50 \) is not a thin-wire case and thus it is an approximation to a thin-wire loop.

The first case is a loop antenna with a very thin wire with \( b = 50 \text{ mm} \) and \( a = 0.779 \text{ mm} \), \( g = a \). This case corresponds to Table III in the Storer's paper where \( 2\pi b/a = 403.43 \). Figure 3 shows the geometry of the loop antenna.

In Table 1, formulas (5) and (8) are compared with the Storer's theory and the CST numerical simulation for \( kb \) from 0.025 to 0.4. The parameter \( kb \) is the loop perimeter in terms of wavelength. In an approximate model for a loop antenna, the loop is treated as a shorted transmission line of length \( ab \) and a characteristic impedance which is a function of the loop radius \( b \) and the wire radius \( a \).

![Figure 3. Geometry of a loop antenna with \( b = 50 \text{ mm} \) and \( a = 0.779 \text{ mm} \)](image)

**Table 1. Input impedance (\( \Omega \)) of a loop with \( b = 50 \text{ mm} \) and \( a = 0.779 \text{ mm} \)**

<table>
<thead>
<tr>
<th>Method</th>
<th>( kb )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.025</td>
</tr>
<tr>
<td>Form-1</td>
<td>7.71×10^{-2}+j357.3</td>
</tr>
<tr>
<td>Form-2</td>
<td>8.26×10^{-2}+j130</td>
</tr>
<tr>
<td>CST-F</td>
<td>9.06×10^{-4}+j40.0</td>
</tr>
<tr>
<td>Storer</td>
<td>-</td>
</tr>
</tbody>
</table>

In Table 1, first we observe good agreements between the Storer's theory and the simulation. The formula (5) (Form-1) provides useful approximate values of \( R \) and \( X \), only for \( kb < 0.2 \). The formula (8) (Form-2) gives reasonable values of \( R \) for \( kb < 0.4 \) while its \( X \) values is two to three times larger than accurate values. The reason for this large discrepancy seems to lie in the fact that the modeling of a circular loop antenna with a uniform two-wire transmission line is too crude. Also one can notice that the accuracy of formula (5) is poor even for small loop cases \( kb = 0.025 \) and 0.05. The reason for the inaccuracy is due to the fact that the loop current can never be uniform due to the presence of the excitation in an infinitesimal or finite gap which destroys the circular symmetry.

Figure 4 shows the input impedance versus frequency of the loop antenna shown in Figure 2. The \( kb \) values of 0.025, 0.05, 0.10, 0.20, 0.30, and 0.40 correspond to 0.023875, 0.04775, 0.0955, 0.191, 0.2865, 0.382 GHz, respectively. The antiresonance frequency is 0.4253 GHz. A small change in the antiresonance frequency results in a large change in...
the input impedance of the loop antenna at frequencies near the antiresonance frequency. Therefore any formula which does not model the antiresonance frequency accurately will not provide accurate impedance values for \( kb > 0.35 \).

Figure 5 shows a semi-logarithmic plot of the input impedance calculated by CST-F of the loop antenna of Figure 1 from 0.01 to 0.4 GHz. CST-F denotes the Frequency Domain Solver of the CST Studio Suite. It is interesting to note that \( \log R \) and \( \log X \) can be approximated with a linear function of the frequency in some intervals. In the frequency range 0.01 to 0.02 GHz, there is a dip in the simulated values of \( R \) due to the well-known low-frequency stability problem in the frequency-domain numerical methods.

The second case is a loop antenna with a moderately thin wire with \( b = 50 \) mm and \( a = 2.11 \) mm, \( g = a \). This case corresponds to Table II in the Storer's paper where \( 2yb/a = 148.41 \). Figure 6 shows the geometry of the loop antenna. From the figure, it appears that the loop can still be treated as a thin-wire antenna.

Table 2 shows a comparison of formulas (5) and (8) for the input impedance of the loop antenna shown in Figure 4. The formula (5) provides useful values of \( R \) for \( kb < 0.2 \) and \( X \) for \( kb < 0.3 \). As in the first loop, the formula (8) gives significantly larger values of \( X \) for all ranges of \( kb \). \( R \) values by the formula (8) are useful for \( kb < 0.4 \).

Figure 7 shows the input impedance versus frequency of the loop antenna shown in Figure 6. The antiresonance frequency is 0.4135 GHz with a reduction by 0.0118 GHz from that of the loop with \( a = 0.779 \) mm. The frequency-bandwidth sharpness of the antenna is reduced from the case with a thinner wire.
Table 2. Input impedance (Ω) of a loop with \(b\) = 50 mm and \(a\) = 2.11 m

<table>
<thead>
<tr>
<th>Method</th>
<th>(kb)</th>
<th>0.025</th>
<th>0.05</th>
<th>0.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Form-1</td>
<td>7.71 \times 10^{-5} + j47.9</td>
<td>0.00123 + j95.7</td>
<td>0.0197 + j191</td>
<td></td>
</tr>
<tr>
<td>Form-2</td>
<td>8.26 \times 10^{-5} + j109</td>
<td>0.00129 + j219</td>
<td>0.0217 + j448</td>
<td></td>
</tr>
<tr>
<td>CST-F</td>
<td>5.62 \times 10^{-5} + j30.6</td>
<td>0.00845 + j61.7</td>
<td>0.0379 + j128</td>
<td></td>
</tr>
<tr>
<td>Storer</td>
<td>-</td>
<td>0.0051 + j62.6</td>
<td>0.0410 + j128</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Method</th>
<th>(kb)</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Form-1</td>
<td>0.316 + j382</td>
<td>1.60 + j574</td>
<td>5.05 + j766</td>
<td></td>
</tr>
<tr>
<td>Form-2</td>
<td>0.511 + j1002</td>
<td>5.57 + j1899</td>
<td>107 + j4247</td>
<td></td>
</tr>
<tr>
<td>CST-F</td>
<td>0.470 + j298</td>
<td>6.49 + j628</td>
<td>162 + j2123</td>
<td></td>
</tr>
<tr>
<td>Storer</td>
<td>0.594 + j298</td>
<td>6.36 + j624</td>
<td>160 + j2063</td>
<td></td>
</tr>
</tbody>
</table>

The final case is a thick-wire loop shown in Figure 8 where \(b\) = 50 mm, \(a\) = 5.75 mm, and \(g = a\). For this case, the thin-wire model of the formulas (5) and (8) and Storer is only an approximation.

Table 3 shows the input impedance of the loop antenna of Figure 8. The formulas (5) and (8) show similar accuracy as in the previous cases. Figure 9 shows the input impedance versus frequency of the loop antenna of Figure 8. The antiresonance frequency is 0.4068 GHz, which is reduced by 0.0185 GHz from that of the loop with \(a = 0.779\) mm. The frequency-bandwidth sharpness of the antenna is reduced from the case of a thin wire.
III. CONCLUSION

Simple formulas for the input impedance of a small loop antenna have been tested for their accuracy against the numerical simulation and the theory. It has been observed that the textbook formula as well as the formula by Adawalla and Sharshar give values significantly different from accurate values for the loops whose perimeter is less than 0.2 or 0.3 wavelength. Thus these formulas might be useful for rough first-pass design works. For good accuracy, the theoretical values or the numerical simulation is recommended.

REFERENCES


BIOGRAPHIES

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