Analytic Integration of the Free-Space Green’s Function over a Rectangular Region

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Abstract

This paper presents the analytic integration of the free-space Green’s function over rectangular regions arising in electromagnetic theory. Specifically, closed-form formulas are presented for the integration of $1/R$ and $1/R^2$ functions over a linear segment, over a rectangular surface, and over a rectangular volume. Presented formulas can be used for the calculation of the electrostatic potential, the electric field strength, the magnetic vector potential, and the magnetic field strength due to source (charge or current) uniformly distributed over rectangular regions.

Key words: Analytic integration, Free-space Green’s function, Rectangular region, Electromagnetics

I. Introduction

The Green’s function offers a powerful tool applicable to a wide range of problems in engineering and science, where the calculation of the field or effect due to a prescribed source or excitation is required. The cause and effect of many physical phenomena depends on $1/R$ or $R^2/R$, where $R$ is the distance vector from a source point to a field point and $R$ is its magnitude. Many types of the Green’s function are of $1/R$ or $R^2/R$ type. Sources of arbitrary distribution in a region of general shape can be decomposed into uniformly distributed sources over small rectangular regions, the fields of which are summed to find the total field. Therefore the analytic integration of $1/R$ and $R^2/R$ function over rectangular regions is of fundamental importance.

The close-form or analytic integration of the free-space Green’s function is also important in the computation of the elements of a coefficient matrix in the method of moments or in the boundary element method. Some closed-form formulas are available for the integration of $1/R$ and $R^2/R$ over rectangular regions [1]-[8]. A comprehensive set of closed-form formulas for the calculation of the electric scalar potential, for the electrostatic field, for the magnetic vector potential, and for the magnetic field due to a source in rectangular regions has not been published in open literature. In this paper we derive formulas for the integration of $1/R$ and $R^2/R$ over a linear segment, over a rectangular surface, and over a rectangular volume. Potential and field functions due to sources of rectangular shape are expressed using derived formulas.

II. Analytic Integration

Fig. 1 shows a linear segment, a rectangular surface, and a rectangular volume over which charge or current is uniformly distributed. The local basis vectors are set up in the source region to define a point $(u, v, w)$ in space.

![Fig. 1. A linear segment (a), a rectangular surface (b), and a rectangular volume (c) with a local rectangular coordinate vectors $\hat{u}, \hat{v}, \hat{w}$.](image-url)
A vector $\mathbf{R}$ from a source point $\mathbf{r}$ and to a field point $\mathbf{r}'$ can be written in the rectangular coordinate system as

$$\mathbf{R} = \mathbf{r}' - \mathbf{r} = (u' - u)\hat{u} + (v' - v)\hat{v} + (w' - w)\hat{w}$$  \hspace{1cm} (1)

where a field point is defined by

$$\hat{u} \hat{v} \hat{w} = + + + \mathbf{r} \hat{u} \hat{v} \hat{w}$$  \hspace{1cm} (2a)

and a source point by

$$\hat{u} \hat{v} \hat{w} = - = = \mathbf{r} \hat{u} \hat{v} \hat{w}$$  \hspace{1cm} (2b)

for sources defined over a linear segment, over a rectangular surface, and over a rectangular volume, respectively. The usual convention of denoting the source and field points with $\mathbf{r}'$ and $\mathbf{r}$ respectively is reversed to drop the prime symbol in integral formulas. The magnitude of $\mathbf{R}$ is given by

$$R = [(u' - u)^2 + (v' - v)^2 + (w' - w)^2]^{1/2}$$  \hspace{1cm} (2c)

In integration formulas, we omit $\hat{u}, \hat{v}, \hat{w}$ and we use $u \rightarrow u - u', \ v \rightarrow v - v', \ w \rightarrow w - w'$ (4)

when applying formulas in the actual calculation of the integral. In the following we will use relations

$$\mathbf{R} = \mathbf{r}' - \mathbf{r} = -u'\hat{u} - v'\hat{v} - w'\hat{w}$$  \hspace{1cm} (5a)

$$R = [u'^2 + v'^2 + w'^2]^{1/2}$$  \hspace{1cm} (5b)

$$\frac{\partial R}{\partial u} = - u, \frac{\partial R}{\partial v} = - v, \frac{\partial R}{\partial w} = - w R$$  \hspace{1cm} (5c)

$$-\nabla f(R) = - \frac{df}{dR} \left( \frac{du}{du'} \hat{u} + \frac{dv}{dv'} \hat{v} + \frac{dw}{dw'} \hat{w} \right)$$

$$= - \frac{df}{dR} \left( \frac{u'}{R} \hat{u} + \frac{v'}{R} \hat{v} + \frac{w'}{R} \hat{w} \right)$$

$$= \frac{df}{dR} (-\mathbf{R}) = \nabla f(R)$$  \hspace{1cm} (5d)

The electric potential due to charge uniformly distributed over a linear segment, over a rectangular surface, and over a rectangular volume can be written as

$$V = \frac{\rho_1}{4\pi\varepsilon} \int \frac{1}{R} \, dw$$  \hspace{1cm} (6a)

$$V = \frac{\rho_2}{4\pi\varepsilon} \int \frac{1}{R} \, du dv$$  \hspace{1cm} (6b)

$$V = \frac{\rho_3}{4\pi\varepsilon} \int \frac{1}{R} \, du dv dw$$  \hspace{1cm} (6c)

where $\rho_1, \rho_2, \rho_3$ are charge densities in one, two, and three dimensions respectively, and $\varepsilon$ is the permittivity of the medium. We define

$$K_1 = \int \frac{1}{R} \, dw$$  \hspace{1cm} (7a)

$$K_2 = \int \frac{1}{R} \, du dv$$  \hspace{1cm} (7b)

$$K_3 = \int \frac{1}{R} \, du dv dw$$  \hspace{1cm} (7c)

Then (6) can be written as

$$V = \frac{\rho_1}{4\pi\varepsilon} K_1; \ V = \frac{\rho_2}{4\pi\varepsilon} K_2; \ V = \frac{\rho_3}{4\pi\varepsilon} K_3$$  \hspace{1cm} (8)

The magnetic vector potential due to current uniformly flowing on a linear segment, on a rectangular surface, and through a rectangular volume can be written as

$$\mathbf{A} = \frac{\mu_0 I}{4\pi} \int \frac{1}{R} \, dw = \frac{\mu_0 I}{4\pi} K_1$$  \hspace{1cm} (9a)

$$\mathbf{A} = \frac{\mu_0 J_s}{4\pi} \int \frac{1}{R} \, du dv = \frac{\mu_0 J_s}{4\pi} K_2$$  \hspace{1cm} (9b)

$$\mathbf{A} = \frac{\mu_0 J_v}{4\pi} \int \frac{1}{R} \, du dv dw = \frac{\mu_0 J_v}{4\pi} K_3$$  \hspace{1cm} (9c)

where $\mathbf{I}$ is current vector on a linear segment, $\mathbf{J}_s$ is surface current density on a rectangular surface, and $\mathbf{J}_v$ is volume current density in a rectangular volume. The electric field $\mathbf{E}$ due to charge uniformly distributed over a linear segment, over a rectangular surface, and over a rectangular volume can be written as

$$\mathbf{E} = \frac{\rho_1}{4\pi\varepsilon} \int \frac{\mathbf{R}}{R^3} \, dw$$  \hspace{1cm} (10a)

$$\mathbf{E} = \frac{\rho_2}{4\pi\varepsilon} \int \frac{\mathbf{R}}{R^3} \, du dv$$  \hspace{1cm} (10b)

$$\mathbf{E} = \frac{\rho_3}{4\pi\varepsilon} \int \frac{\mathbf{R}}{R^3} \, du dv dw$$  \hspace{1cm} (10c)

$$\mathbf{E}_L = \int \frac{\mathbf{R}}{R^3} \, dw$$  \hspace{1cm} (11a)

$$\mathbf{E}_L = \int \frac{\mathbf{R}}{R^3} \, du dv$$  \hspace{1cm} (11b)

$$\mathbf{E}_L = \int \frac{\mathbf{R}}{R^3} \, du dv dw$$  \hspace{1cm} (11c)

Then (10) can be written as

$$\mathbf{E} = \frac{\rho_1}{4\pi\varepsilon} \mathbf{L}_1; \ \mathbf{E} = \frac{\rho_2}{4\pi\varepsilon} \mathbf{L}_2; \ \mathbf{E} = \frac{\rho_3}{4\pi\varepsilon} \mathbf{L}_3$$  \hspace{1cm} (12)

The magnetic field $\mathbf{H}$ due to current uniformly flowing on a linear segment, on a rectangular surface, and through a rectangular volume can be written as

$$\mathbf{H} = \frac{\mu_0 I}{4\pi} \int \frac{\mathbf{R}}{R^3} \, dw = \frac{\mu_0 I}{4\pi} \mathbf{L}_1$$  \hspace{1cm} (13a)

$$\mathbf{H} = \frac{\mu_0 J_s}{4\pi} \int \frac{\mathbf{R}}{R^3} \, du dv = \frac{\mu_0 J_s}{4\pi} \mathbf{L}_2$$  \hspace{1cm} (13b)

$$\mathbf{H} = \frac{\mu_0 J_v}{4\pi} \int \frac{\mathbf{R}}{R^3} \, du dv dw = \frac{\mu_0 J_v}{4\pi} \mathbf{L}_3$$  \hspace{1cm} (13c)
Now we consider the analytic evaluation of integrals of (7) and (11). First, the integral \( K_0 \) is given in a standard handbook of integral, which can be derived by substitution method:

\[
K_1 = \int \frac{1}{R} \, dw = \int \frac{1}{R + w} \, dw = \int \frac{1}{R + w} \left( 1 + \frac{w}{R} \right) \, dw
\]

We use

\[
t = R + w, \quad dt = \left( \frac{w}{R} + 1 \right) \, dw
\]

to obtain

\[
K_1 = \int \frac{1}{t} \, dt = \log t = \log(R + w)
\]

The integral \( K_1 \) can be expressed in many forms [1]:

\[
K_1 = \int \frac{1}{R + w} \, dw = \log(R + w) = \log \left( \frac{R + w}{R - w} \right)
\]

In the following we will use

\[
K_1 = \int \frac{1}{R} \, dw = \log(R + w)
\]

Next we evaluate the integral \( K_2 \).

\[
K_2 = \int \frac{1}{R} \, dv = \log(R + w)
\]

Integrating (16) by parts yields

\[
K_2 = v \log(R + u) - \int \frac{1}{R + u} \, dv
\]

To find the second term in (17a), algebraic manipulation is required:

\[
\frac{1}{R + u} \, dv = \frac{v^2}{R + u} \, dv = \frac{(v^2 + w^2)(R - u) - u^2(R - u)}{R(v^2 + w^2)} = \frac{R - u - u w^2}{R - v^2 + w^2} + \frac{u w^2}{R(v^2 + w^2)}
\]

\[
= 1 - u \frac{w^2}{v^2 + w^2} + \frac{w^2 u}{R(v^2 + w^2)}
\]

Using (17d) in (17b), we obtain

\[
K_2 = \int \log(R + u) \, dv = v \log(R + u) + u \log(R + v) - \frac{w \tan^{-1} \frac{uv}{wR}}{w}
\]

The last two terms are constant with respect to the variable \( u \) so that they do not contribute to the definite integral with respect to \( u \). Thus

\[
K_2 = \int \log(R + u) \, dv = v \log(R + u) + u \log(R + v) - \frac{w \tan^{-1} \frac{uv}{wR}}{w}
\]

Using the inverse hyperbolic tangent in (17b), we finally obtain

\[
K_2 = \int \log(R + u) \, dv = v \tan^{-1} \frac{u}{R} + u \tan^{-1} \frac{v}{R} - \frac{w \tan^{-1} \frac{uv}{wR}}{w}
\]

next we derive a formula for the integral \( K_3 \):

\[
K_3 = \int \int \frac{1}{R} \, dv \, dw
\]

\[
= \int \left[ u \ln(R + v) + v \ln(R + u) - w \tan^{-1} \frac{uv}{wR} \right] \, dv
\]

The integration of the first two terms in (18a) can be done using (17i):
The integration of the third term of (18a) requires algebraic manipulation:

\[
    K_{3a} = \int w \log(R + v) \, dw = \left[ w \tan^{-1} \frac{v}{w} + v \tan^{-1} \frac{w}{R} - u \tan^{-1} \frac{uv}{uR} \right] \tag{18b}
\]

\[
    K_{3b} = \int v \log(R + u) \, dw = \left[ v \tan^{-1} \frac{u}{w} + u \tan^{-1} \frac{w}{R} - v \tan^{-1} \frac{wu}{vR} \right] \tag{18c}
\]

The integration of the third term of (18a) requires algebraic manipulation:

\[
    K_{3c} = \int w \tan^{-1} \frac{w}{R} \, dw = \frac{w^2}{2} \tan^{-1} \frac{w}{uR} + \frac{1}{2} \int \frac{1}{w^2 - u^2} \, dw \]

\[
    = \frac{w^2}{2} \tan^{-1} \frac{w}{uR} + \frac{1}{2} \left( \frac{1}{w^2 - u^2} \right) \left( uR - w \right) \tag{18d}
\]

where the following relation is used.

\[
    \frac{\partial}{\partial w} \tan^{-1} \frac{w}{R} = \frac{w}{u + (wu/wR)^2} \left( \frac{-1}{w^2 + u^2} + \frac{1}{wu} \right) \]

\[
    = -\frac{w}{u^2 + v^2} \left( \frac{1}{w^2 + u^2} \right) \left( R + \frac{w^2}{R} \right) \]

\[
    = -\frac{uv}{u^2 + v^2} \left( \frac{w^2 + u^2}{R} \right) \left( w^2 + u^2 \right) \]

\[
    = -\frac{uv}{R} \left( \frac{1}{w^2 + u^2} + \frac{1}{w^2 + u^2} \right) \tag{18e}
\]

From (18a)-(18d), we obtain

\[
    K = uv \tan^{-1} \frac{w}{R} + vw \tan^{-1} \frac{u}{w} + u \tan^{-1} \frac{uv}{uR} \]

\[
    = \frac{u^2 v}{2} \tan^{-1} \frac{w}{uR} - \frac{v^2}{2} \tan^{-1} \frac{w}{vR} - \frac{w^2}{2} \tan^{-1} \frac{uv}{wR} \tag{19a}
\]

Using

\[
    \int \frac{1}{R^3} \, dw = \frac{1}{(u^2 + v^2)R} \tag{19b}
\]

\[
    \int \frac{w^3}{R^3} \, dw = -\frac{1}{R} \tag{19c}
\]

\[
    L_1 = -\left( u \hat{v} + v \hat{w} \right) w + \frac{1}{R} \hat{w} \tag{19d}
\]

Formula (19d) can also be obtained from \( K \) by differentiation:

\[
    L_1 = \nabla K_1
\]

\[
    = \frac{1}{R + wR} + \frac{1}{R + wR} \hat{v} \left( 1 + \frac{w}{R} \right) \hat{w} \]

\[
    = \frac{R - w}{u^2 + v^2} + \frac{R - w}{u^2 + v^2} \hat{v} \left( 1 + \frac{w}{R} \right) \hat{w} \]

\[
    = -\left( u \hat{v} + v \hat{w} \right) w + \frac{1}{R} \hat{w} + u \hat{u} + v \hat{v} \tag{19e}
\]

The last term in (19e) can be dropped since it is constant with respect to \( w \).

Next consider the evaluation of \( L_2 \).

We denote the last term as \( L_{2a} \). Using (19b), we obtain

\[
    L_{2a} = \int \frac{w}{R^3} \, dw = \int \frac{w^4}{(v^2 + w^2)^2} R \tag{20a}
\]

Using

\[
    \frac{wu}{(v^2 + w^2)^2} R = \frac{wu}{(v^2 + w^2)^2} \left( v^2 + w^2 \right) \left( v^2 + w^2 \right) \]

\[
    = \frac{wu}{(v^2 + w^2)^2} \left( v^2 + w^2 \right) \tag{20b}
\]

in (20b) gives

\[
    L_{2a} = \int \frac{w}{R^3} \, dw = \int \frac{w^4}{(v^2 + w^2)^2} R \tag{20c}
\]

\[
    L_2 = \nabla L_2
\]

\[
    = \int \frac{\hat{u}}{R} + \hat{v} \left( 1 + \frac{w}{R} \right) \hat{w} \int \frac{w}{R^3} \, dw \tag{20d}
\]

Formula (20d) can also be obtained by differentiation:

\[
    L_2 = \nabla L_2
\]

Finally we derive a formula for \( L_3 \).
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\[
L_3 = \iiint \frac{-u \hat{u} - v \hat{v} - w \hat{w}}{R^3} \, dudvdw
 = \hat{u} \iint \frac{1}{R} \, dudw + \hat{v} \iint \frac{1}{R} \, dvdu + \hat{w} \iint \frac{1}{R} \, dwdv
\]

(21a)

Using (17g) in (21a), we obtain

\[
L_3 = \hat{u} \left[ v \tanh^{-1} \frac{w}{R} + w \tanh^{-1} \frac{v}{R} - u \tanh^{-1} \frac{wv}{uR} \right] + \\
\hat{v} \left[ w \tanh^{-1} \frac{1}{R} + u \tanh^{-1} \frac{w}{R} - v \tanh^{-1} \frac{wR}{vR} \right] + \\
\hat{w} \left[ u \tanh^{-1} \frac{v}{R} + v \tanh^{-1} \frac{1}{R} - w \tanh^{-1} \frac{uv}{wR} \right]
\]

(21b)

Formula (21b) can also be obtained by differentiation of \(K_3\):

\[
L_3 = \nabla K_3
\]

(21c)

This completes the derivation of analytic integration formulas of \(1/R\) and \(R/R^3\) for rectangular regions.

### III. Conclusion

Analytic integration formulas have been derived for \(1/R\) and \(R/R^3\) for rectangular regions such as a linear segment, a rectangular surface, and a rectangular volume. Derived formulas can be applied to the calculation of the electric scalar potential, the static electric field, the magnetic vector potential, and the static magnetic field due to sources uniformly distributed over rectangular regions. Derived formulas can also be used in the calculation of electric or magnetic fields due to arbitrary distribution of charge or current by pulse-type (piecewise constant) approximation. They can also be used in the calculation of the matrix elements arising in the method of moment or in the boundary element method.

### References


