Capacitance of Parallel Cylinders of Unequal Radii By Image Method

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Abstract

The structure of two parallel cylinders or wires is of fundamental importance in signal and power transmission in electrical engineering fields. In this paper, the problem of finding the capacitance of two parallel cylinders of unequal radii is investigated. First, existing works on the problem are reviewed. An infinitely long cylinder of constant potential is analyzed using the image theory, which is then applied to the evaluation of the capacitance of two parallel cylinders of unequal radii. Two cylinders may be separated from each other or one cylinder can be enclosed by the other. It is shown that the capacitance formula is reduced to the well-known formulas for a two-wire transmission line or to a coaxial cable.

Keywords: Capacitance, electromagnetics, two-wire transmission line, image theory

I. INTRODUCTION

The evaluation of static capacitance of two dimensional structures is important in signal transmission and signal integrity applications. The capacitance of a two-wire structure of equal radii is derived in many undergraduate textbooks, for example in Hayt and Buck [1]. Although the mathematics involved is the same, the derivation of Hayt and Buck is rather lengthy that students can easily lose their way in a maze of equations. Green [2] derived a capacitance formula for two wires of equal radii from a geometrical analysis, which is simpler than the Hayt-Buck derivation and is not easily extendable to the problem of two wires of different radii.

In the case of two wires of unequal radii, earlier researchers, for example Smythe [3], employed the complex variable theory to obtain a capacitance formula. In [4], capacitance formulas for various two dimensional geometries are given including two parallel wires of unequal radii. Das and co-workers [5] have presented the analytical method for the capacitance of dielectric coated two parallel wires of unequal radii. Their formulas involve integration of an expression involving transcendental functions, which was numerically done by the authors.

In this paper, the capacitance formulas of two parallel wires of unequal radii are derived using the image theory for the electrostatic two dimensional cylinder problem. First a cylinder of constant potential is represented by a line charge of positive polarity and a corresponding image line charge. Then the results are applied to the two parallel wire problem and the capacitance formula is derived. It is shown that the derived formula is reduced to standard textbook formulas for two wires of equal radii.

II. IMAGE THEORY

Assume that two uniformly charged infinitely long line charges are placed at \( r_1 \) and \( r_2 \) on the \( xy \)-plane. The electric potential at a field point \( r \) is given by

\[
V = \frac{q_1}{2\pi \varepsilon} \ln \left| \frac{r-r_1}{r-r_2} \right| + \frac{q_2}{2\pi \varepsilon} \ln \left| \frac{r-r_1}{r-r_2} \right| \tag{1}
\]

where \( r_1 \) is a reference point for the electric potential, \( q_1 \) and \( q_2 \) are the charge density per unit length corresponding to lines sources at \( r_1 \) and \( r_2 \). If we let

\[
q_1 = -q_2 = q
\]

then

\[
V = \frac{q}{2\pi \varepsilon} \ln \left| \frac{r-r_2}{r-r_1} \right| \tag{3}
\]

If we let

\[
|r-r_2| = k|r-r_1|, \quad k : \text{constant} \tag{4}
\]

then the geometry defined by (4) is an equipotential surface. Equation (4) describes a circle whose parameters are given by

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The potential on the cylinder is given by

\[ V_a = \frac{q}{2\pi l} \ln \frac{D-a}{a} = \frac{q}{2\pi l} \ln \frac{D}{a} \]  

(8)

\[ V_b = -\frac{q}{2\pi l} \ln \frac{D-a}{b} \]  

(9b)

Now we apply the above result to the structure of two parallel wires of unequal radii shown in Fig. 2. The potential on each cylinder is given by

\[ V_a = \frac{q}{2\pi l} \ln \frac{D-a}{a} \]  

(9a)

\[ V_b = -\frac{q}{2\pi l} \ln \frac{D-a}{b} \]

(9b)

where \( r_0 \) is the center of the circle, \( R \) is the radius, \( r_a \) and \( r_b \) are the end points of the diameter.

III. CAPACITANCE FORMULA

When (4) is applied to the case shown in Fig. 1, where an equipotential cylinder of radius \( a \) with \( V = V_a \) is obtained by two line sources with \(-q\) at a distance \( D \) from the cylinder center \( O \), and \(+q\) at a distance \( a \) from \( O \), we obtain

\[ a - \alpha = k(D - a) \]  

(6a)

\[ a + \alpha = k(D + a) \]  

(6b)

By Gauss theorem, we find that the total charge per unit length on each cylinder is given by

\[ Q_a = q, Q_b = -q \]  

(11)

The potential difference between two cylinders is given by

\[ \Delta V = V_a - V_b = \frac{q}{2\pi l} \ln \frac{(D-a)(D-\beta)}{ab} \]  

(12)

The capacitance between two cylinders is now given by

\[ C = \frac{Q_a}{\Delta V} = \frac{2\pi l}{\ln \frac{(D-a)(D-\beta)}{ab}} \]  

(13)

In (13), we need to express \( \alpha \) and \( \beta \) in terms of \( a, b, \) and \( D \). To this end, we consider

\[ (D-a)(D-\beta) = D^2-(\alpha+\beta)D+\alpha\beta \]  

(14)

From (10a) and (10b), we obtain

\[ -(\alpha+\beta)D = -2\alpha\beta-(a^2+b^2) \]  

(15)

From (14) and (15), we obtain

\[ (D-a)(D-\beta) = D^2-(\alpha^2+b^2) - \alpha\beta \]  

(16)

Multiplying (10a) with (10b) gives

\[ \alpha\beta[(\alpha+\beta)D^2 - (\alpha+\beta)D] = a^2b^2 \]  

(17)

By combining (17) with (15), we obtain a quadratic equation for \( \alpha\beta \),

\[ (\alpha\beta)^2 + (a^2 + b^2 - D^2)\alpha\beta + a^2b^2 = 0 \]  

(18)
the solution of which is
\[ \alpha \beta = -(a^2 + b^2 - D^2) \pm \sqrt{(a^2 + b^2 - D^2)^2 - 4a^2 b^2} \]  \hspace{1cm} (19)

Since
\[ \alpha \beta \to 0 \hspace{0.2cm} \text{as} \hspace{0.2cm} a^2 + b^2 \to 0 \]  \hspace{1cm} (20)

the valid solution of \( \alpha \beta \) is given by
\[ \alpha \beta = -\frac{(a^2 + b^2 - D^2) - \sqrt{(a^2 + b^2 - D^2)^2 - 4a^2 b^2}}{2} \]  \hspace{1cm} (21)

Using (21) and (16), we finally obtain
\[ \ln \left( \frac{D - \alpha}{D - \beta} \right) \]  \hspace{1cm} (22)
\[ = \ln \left( \frac{D^2 - a^2 - b^2}{2ab} + \sqrt{(D^2 - a^2 - b^2)^2 - 4a^2 b^2} \right) \]  \hspace{1cm} (23)
\[ = \cosh^{-1} \left( \frac{D^2 - a^2 - b^2}{2ab} \right) \]

where the identity
\[ \ln(x + \sqrt{x^2 - 1}) = \cosh^{-1} x \]  \hspace{1cm} (24)

is used.

The final expression for the capacitance of two parallel wires of unequal radii is given by
\[ C = \frac{2\pi \epsilon}{\cosh^{-1} \left( \frac{D^2 - a^2 - b^2}{2ab} \right)} \]  \hspace{1cm} (25)

which is same as the formula by Smythe [3] obtained using the complex variable theory. For a symmetrical case \((a = b)\), the standard textbook formula reads
\[ C = \frac{\pi \epsilon}{\cosh^{-1} \left( \frac{D}{2a} \right)} \]  \hspace{1cm} (26)

whereas (24) gives
\[ C = \frac{\pi \epsilon}{\cosh^{-1} \left( \frac{D^2 - 2a^2}{2a^2} \right)} \]  \hspace{1cm} (27)

Using (23) it is easy to show that
\[ \frac{1}{2} \cosh^{-1} (2x^2 - 1) = \cosh^{-1} x \]  \hspace{1cm} (28)

Thus (26) is equivalent to (25).

To calculate the voltage on the cylinder, one find \( \alpha \) and \( \beta \) using (10a) and (10b).
\[ \alpha = \frac{a^2 - b^2 + D^2 - \sqrt{(a^2 - b^2 + D^2)^2 - 4a^2 b^2}}{2D} \]  \hspace{1cm} (28a)
\[ \beta = \frac{b^2 - a^2 + D^2 - \sqrt{(b^2 - a^2 + D^2)^2 - 4b^2 D^2}}{2D} \]  \hspace{1cm} (28b)

The values of \( \alpha \) and \( \beta \) can also be found by an iterative method. First, place a point charge \( -q \) at \( \alpha \) distance to the right of the left cylinder's center. The initial value of \( \alpha \) is zero. Next find the location \( \beta \) of the image charge \( -q \) to the left of the right cylinder's center, which is given by
\[ \beta = \frac{b^2}{D - \alpha} \]  \hspace{1cm} (29a)

With an image charge \( -q \), find a new value of \( \alpha \) given by
\[ \alpha = \frac{a^2}{D - \beta} \]  \hspace{1cm} (29b)

With this process, \( \alpha \) and \( \beta \) converge fast. The following is a pseudocode and a sample run for finding \( \alpha \) and \( \beta \) when \( a = 1 \text{ m, } b = 2 \text{ m, and } h = 5 \text{ m.} \)

\[
\text{eps}=1e-6
\]
\[
\text{WRITE} \ 'a,b,d(m)='; \\
\text{READ} \ a,b,d
\]
\[
\alpha_{\text{old}}=0; \ \alpha=\alpha_{\text{old}} \\
\beta_{\text{old}}=0; \ \beta=\beta_{\text{old}} \\
i=1
\]

\[
\text{REPEAT}
\]
\[
\text{IF} \ d < a+b \hspace{2cm} \beta=b**2/(d-alpha) \hspace{2cm} \alpha=a**2/(d-beta) \\
\text{ELSE IF} \ d < \text{ABS}(a-b) \hspace{2cm} \beta=b**2/(d+alpha) \hspace{2cm} \alpha=a**2/(beta-d) \\
\text{END IF}
\]
\[
\text{WRITE} \ 'i,\alpha,\beta,=\,'i,\alpha,\beta
\]
\[
\text{IF} \ (\text{ABS}(\alpha-\alpha_{\text{old}}) < \text{eps}) \ \text{AND} \ \ (\text{ABS}(\beta-\beta_{\text{old}}) < \text{eps}) \\
\text{STOP REPEAT}
\]
ELSE  
   alpha_old=alpha; beta_old=beta  
   ADD 1 TO i  
END IF  
END REPEAT  
STOP  

i,alpha,beta= 1 0.238095238095238 0.800000000000000  
i,alpha,beta= 2 0.240384615384615 0.840000000000000  
i,alpha,beta= 3 0.240407965031569 0.840404040404040  
i,alpha,beta= 4 0.240408203316652 0.840408163265306  
i,alpha,beta= 5 0.240408205748385 0.840408205339656  

From (29a) and (29b), we note that  
\[ \alpha \to a, \beta \to b \text{ as } D \to a+b \]  
\[ \alpha \to 0, \beta \to 0 \text{ as } D \to \infty \]  
(30a)  
(30b)  

Next, we consider a case where one cylinder is contained in the other cylinder as shown in Fig. 3. Procedures of deriving the capacitance formulae are similar to those for the case where two wires are separated. The outer cylinder has a radius of \( b \) with its center at \( O' \) while the inner cylinder has a radius of \( a \) with its center at \( O \). To form equipotential surfaces, we place two line charges of density \( +q \) at a distance \( a \) from \( O \) and \( -q \) at a distance \( \beta \) from \( O' \).

\[ V = V_b \]  

![Image](image.png)  
Figure 3. Image theory applied to two parallel cylinders of unequal radii.

Using (7), we obtain charge locations give by  
\[ \alpha = \frac{a^2}{\beta-D} \text{ or } \alpha(\beta-D) = a^2 \]  
\[ \alpha + D = \frac{b^2}{\beta} \text{ or } (\alpha + D)\beta = b^2 \]  
(30a)  
(30b)  

According to (8), the electric potential on each cylinder is given by  
\[ V_a = \frac{q}{2\pi\epsilon} \ln \frac{\beta-D}{\alpha} = \frac{q}{2\pi\epsilon} \ln \frac{a}{\alpha} \]  
(31a)  

By the Gauss theorem, the total charge per unit length on the inner cylinder is given by  
\[ Q_a = q \]  
(31b)  

while on the outer cylinder, we will have  
\[ Q_b = -q \]  
(32a)  

by electrostatic induction.  

The capacitance per unit length is now given by  
\[ C = \frac{Q_a}{\Delta V} = \frac{Q_a}{V_a - V_b} = \frac{2\pi\epsilon}{\ln \frac{a}{b\left(1 + \frac{D}{\alpha}\right)}} \]  
(33)  

In (33), we need to express \( \alpha \) in terms of \( a, b, \) and \( D \). By eliminating \( \beta \) in (30a) and (30b), we find a quadratic equation for \( \alpha \),  
\[ D\alpha^2 + (a^2 - b^2 + D^2)\alpha + a^2D = 0 \]  
(34)  

the solution of which is given by  
\[ \alpha = \frac{b^2 - a^2 - D^2 \pm \sqrt{(b^2 - a^2 - D^2)^2 - 4a^2D^2}}{2D} \]  
(35)  

As \( D \to 0 \), we require \( \alpha \to 0 \) and \( \beta \to \infty \) and thus we have  
\[ \alpha = \frac{b^2 - a^2 - D^2 - \sqrt{(b^2 - a^2 - D^2)^2 - 4a^2D^2}}{2D} \]  
(36a)  
\[ \beta = \frac{a^2}{\alpha} + D \]  
(36b)  

\[ = \frac{b^2 - a^2 + D^2 + \sqrt{(b^2 - a^2 - D^2)^2 - 4a^2D^2}}{2D} \]  
(36c)  

From (30a) and (30b), one observe that  
\[ \alpha \to a, \beta \to b \text{ as } D \to b-a \]  
(37)  

As in the case of the non-overlapping cylinder case, one can find \( \alpha \) and \( \beta \) by an iterative method. The following is the result of a sample run for the case: \( a = 1 \text{ m}, b = 2 \text{ m}, \) and \( h = 0.5 \text{ m} \).  
i,alpha,beta= 1 0.133333333333333 8.00000000000000  
i,alpha,beta= 2 0.171945701357466 6.31578947368421  
i,alpha,beta= 3 0.183389935165174 5.95286195286195  
i,alpha,beta= 4 0.186805072059074 5.85317370679919  
i,alpha,beta= 5 0.187826272598367 5.82406881191921
Using (23) we obtain

\[
\cosh^{-1} \frac{\alpha^2 + b^2}{2ab} = \frac{1}{2} \ln \left( \frac{\alpha^2 + b^2}{2ab} + \sqrt{\left( \frac{\alpha^2 + b^2}{2ab} \right)^2 - 1} \right) = (43)
\]

\[
\ln \left( \frac{\alpha^2 + b^2}{2ab} + \frac{b^2 - d^2}{2ab} \right) = \ln \frac{b}{a}
\]

Thus for a symmetrical case the capacitance formula becomes

\[
C = \frac{2\pi \varepsilon}{\ln \frac{b}{a}}
\]

This completes the derivation of the capacitance formula for two parallel wires of unequal radii.

IV. CONCLUSION

In this paper, the capacitance formula for two parallel wires of unequal radii has been derived using the image theory for an infinite circular cylinder. First, the image theory for the two dimensional cylinder problem has been presented in a general form. Next the image theory has been applied to find the capacitance of two parallel wires of different radii separated by some distance. It is shown that the capacitance formula is reduced to the standard textbook formula when two wires have the same radius. A formula has also been derived for the case where one cylinder is inside the other cylinder. For a symmetrical structure where the center axes of two cylinders coincide, the capacitance formula is reduced to the standard formula of a coaxial cable. The method presented in this paper may be applied to find the capacitance matrix of multiple parallel wire structure, where a sequence of converging image charges are employed.

REFERENCES


BIOGRAPHIES

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